# Law invariant convex risk measures on $\mathbb{R}^d$

W. Schachermayer
joint work in progress with I. Ekeland

University of Vienna
Faculty of Mathematics

January 12, 2011

# The classical one-dimensional case

We extend the following theorems on risk measures, which are well-known for the one-dimensional case, to the *d*-dimensional case.

We work on a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

## Reminder

A function $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is called *law invariant* if $X \sim Y$, i. e. $law(X) = law(Y)$, implies that $\varrho(X) = \varrho(Y)$.

For $F \in L^1_+(\Omega, \mathcal{F}, \mathbb{P})$ normalized by $\mathbb{E}[F] = 1$, we define the law invariant risk measure $\varrho_F : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ by

$$\varrho_F(X) = \sup_{\tilde{X} \sim X} \mathbb{E}[-\tilde{X}F] = \sup_{\tilde{F} \sim F} \mathbb{E}[-X\tilde{F}].$$

The measures $\varrho_F$ have the following co-monotonicity property.

## Definition

A risk measure $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is called co-monotone if, for co-monotone random variables $X$, $Y$, we have

$$\varrho(X+Y) = \varrho(X) + \varrho(Y).$$

Recall that $X$ is co-monotone to $Y$ iff $[X(\omega) - X(\omega')] \cdot [Y(\omega) - Y(\omega')] \geq 0$, for $\mathbb{P} \otimes \mathbb{P}$ almost all $(\omega, \omega') \in \Omega \times \Omega$. We write $X \sim_c Y$.

## Theorem A (Kusuoka, 2001)

For a law invariant convex risk measure $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ t.f.a.e.

(i) $\varrho$ is co-monotone.

(ii) There is $F \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $0 \leq \alpha \leq 1$ s.t.

$$\begin{aligned}\varrho(X) &= \alpha \varrho_F(X) + (1-\alpha) \mathit{ess}\sup(-X) \\ &=: \alpha \varrho_F(X) + (1-\alpha)\varrho^\infty(X).\end{aligned}$$

(iii) $\varrho$ is *strongly cohorent*, i.e.

$$\varrho(X) + \varrho(Y) = \sup_{\tilde{X} \sim X, \tilde{Y} \sim Y} \varrho(\tilde{X} + \tilde{Y}).$$

The equivalence (i) $\Leftrightarrow$ (ii) is due to Kusuoka.
Property (iii) is easily seen to be equivalent to (i) in the one-dimensional setting.
While it is – a priori – not clear how to extend the notion (i) of *co-monotonicity* to the vector-valued setting, the notion (iii) of *strong coherence* extends to the vector-valued case on an obvious way. This is the reason why Ekeland-Galichon-Henry (2009) introduced this notion (in the vector valued setting).

Here is the second theorem which we state for the one-dimensional case, and later extend to the $d$-dimensional one.

## Theorem B (Kusuoka,2001)

Let $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a law invariant convex risk measure. Then there is a function $v : [0,1] \times \mathcal{P} \to [0, \infty]$ such that

$$\varrho(X) = \max_{(\alpha, F) \in [0,1] \times \mathcal{P}} \left\{ \alpha \varrho_F(X) + (1-\alpha)\varrho^\infty(X) - v(\alpha, F) \right\}.$$

## Definition

A convex risk measure on $\mathbb{R}^d$ is a function
$\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ s.t.

(i) (normalization) $\varrho(0) = 0$,,

(ii) (monotonicity) $X \geq Y \Rightarrow \varrho(X) \leq \varrho(Y)$,

(iii) (cash invariance) $\varrho(X + m\mathbb{1}) = \varrho(X) - m$, for $m \in \mathbb{R}$,

(iv) (convexity)

$$\varrho(\alpha X + (1-\alpha)Y) \leq \alpha\varrho(X) + (1-\alpha)\varrho(Y), \qquad \text{for } 0 < \alpha < 1.$$

We call $\varrho$ *coherent* if, in addition,

(v) (positive homogeneity) $\varrho(\lambda X) = \lambda\varrho(X)$, for $\lambda \geq 0$.

On the standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we now denote by $\mathcal{P}$ the subset of $L^1_+(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ of normalized functions $F$ taking their values in $\mathbb{R}^d_+$,

$$\mathcal{P} = \left\{ F = (F_i)_{i=1}^d \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) : F_i \geq 0, \mathbb{E}\left[\sum_{i=1}^d F_i\right] = 1 \right\}.$$

## Definition (Rüschendorf)

For given $F \in \mathcal{P}$ we define $\varrho_F$, the *maximal correlation risk measure in the direction* $F$, by

$$\varrho_F(X) = \sup_{\tilde{X} \sim X} \mathbb{E}\left[-(\tilde{X}, F)\right] = \sup_{\tilde{F} \sim F} \mathbb{E}\left[-(X, \tilde{F})\right].$$

Note that $\varrho_F$ only depends on the law of $F$.

## Proposition (Rüschendorf)

A coherent (resp. convex) risk measure $\varrho : L_d^\infty \to \mathbb{R}$ is *law invariant* if and only if it can be represented as

$$\varrho(X) = \sup_{F \in C} \varrho_F(X)$$

$$\text{resp. } \varrho(X) = \sup_{F \in C} \{\varrho_F(x) - v(F)\}$$

where $C$ is a subset of $\mathcal{P}$ and $v : C \to \mathbb{R}_+$ a non-negative function defined on $C$.

Apart from the risk measures $\varrho_F$, where $F \in \mathcal{P}$, a second type of risk measures will play a special role. It generalizes the maximal loss measure from the one- to the $d$-dimensional case.

## Definition

For $\xi \in S^d$, where

$$S^d := \left\{ \xi \in \mathbb{R}^d : \xi_i \geq 0, \sum_{i=1}^{d} \xi_i = 1 \right\},$$

we define the *maximal loss measure in the direction* $\xi$ by

$$\varrho_\xi^\infty(X) = \mathit{ess}\,\mathrm{sup} \left\{ - \sum_{i=1}^{d} \xi_i X_i \right\}.$$

More generally, for a probability measure $\mu$ on $S^d$ we may define

$$\varrho_\mu^\infty(X) = \int_{S^d} \varrho_\xi^\infty(X) d\mu(\xi).$$

## Theorem (Ekeland, Galichon, Henry, 2009)

Assume that $\varrho$ is a convex, law invariant risk measure on $\mathbb{R}^d$ which extends continuously from $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ to $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, for some $p < \infty$. Then there is a function $v : \mathcal{P} \to [0, \infty]$ such that

$$\varrho(X) = \max_{F \in \mathcal{P}} \{\varrho_F(X) - v(F)\}$$

## Theorem (Ekeland, S., 2010)

Assume that $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ is a convex, law invariant risk measure on $\mathbb{R}^d$. Then there is a function
$v : [0,1] \times \mathcal{P} \times \mathcal{M}_+^1(S^d) \to [0, \infty]$ such that

$$\varrho(X) = \max_{(\alpha, F, \mu) \in [0,1] \times \mathcal{P} \times \mathcal{M}_+^1(S^d)} \left\{\alpha \varrho_F(X) + (1-\alpha)\varrho_\mu^\infty(X) - v(\alpha, F, \mu)\right\}$$

The law invariant risk measure $\varrho$ is coherent if and only if $v$ can be chosen to take only values in $\{0, \infty\}$.

**Definition (Ekeland, Galichon, Henry, 2009)**

A convex, law invariant risk measure $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ is *strongly coherent* if

$$\varrho(X) + \varrho(Y) = \sup \left\{ \varrho(\tilde{X} + \tilde{Y}) : \tilde{X} \sim X, \tilde{Y} \sim Y \right\}.$$

**Theorem (Ekeland, Galichon, Henry, 2009)**

Let $\varrho$ be a convex, law invariant risk measure which extends continuously from $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ to $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, for some $p < \infty$. Then $\varrho$ is strongly coherent if and only if there is some $F \in \mathcal{P}$ such that

$$\varrho(X) = \alpha \varrho_F(X).$$

**Theorem (Ekeland, S., 2010)**

A convex, law invariant risk measure $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ is strongly coherent if and only if there is some $F \in \mathcal{P}, \mu \in \mathcal{M}_+(S^d)$ and $\alpha \in [0, 1]$ such that

$$\varrho(X) = \alpha \varrho_F(X) + (1 - \alpha) \varrho_\mu^\infty(X).$$