Law invariant convex risk measures on $\mathbb{R}^d$

W. Schachermayer
joint work in progress with I. Ekeland

University of Vienna
Faculty of Mathematics

January 12, 2011
We extend the following theorems on risk measures, which are well-known for the one-dimensional case, to the \(d\)-dimensional case.

We work on a standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\).
A function $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$ is called law invariant if $X \sim Y$, i.e. $\text{law}(X) = \text{law}(Y)$, implies that $\varrho(X) = \varrho(Y)$.

For $F \in L^1_+(\Omega, \mathcal{F}, \mathbb{P})$ normalized by $\mathbb{E}[F] = 1$, we define the law invariant risk measure $\varrho_F : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$ by

$$\varrho_F(X) = \sup_{\tilde{X} \sim X} \mathbb{E}[-\tilde{X}F] = \sup_{\tilde{F} \sim F} \mathbb{E}[-X\tilde{F}].$$

The measures $\varrho_F$ have the following co-monotonicity property.
A risk measure $\varphi : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$ is called co-monotone if, for co-monotone random variables $X, Y$, we have

$$\varphi(X + Y) = \varphi(X) + \varphi(Y).$$

Recall that $X$ is co-monotone to $Y$ iff

$$[X(\omega) - X(\omega')] \cdot [Y(\omega) - Y(\omega')] \geq 0,$$

for $\mathbb{P} \otimes \mathbb{P}$ almost all $(\omega, \omega') \in \Omega \times \Omega$. We write $X \sim_c Y$. 

W. Schachermayer joint work in progress with I. Ekeland

Law invariant convex risk measures on $\mathbb{R}^d$
Theorem A (Kusuoka, 2001)

For a law invariant convex risk measure \( \varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \) t.f.a.e.

(i) \( \varrho \) is co-monotone.

(ii) There is \( F \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \) and \( 0 \leq \alpha \leq 1 \) s.t.

\[
\varrho(X) = \alpha \varrho_F(X) + (1 - \alpha) \text{ess sup}(-X) =: \alpha \varrho_F(X) + (1 - \alpha) \varrho^\infty(X).
\]

(iii) \( \varrho \) is strongly coherent, i.e.

\[
\varrho(X) + \varrho(Y) = \sup_{\tilde{X} \sim X, \tilde{Y} \sim Y} \varrho(\tilde{X} + \tilde{Y}).
\]
The equivalence (i) ⇔ (ii) is due to Kusuoka. Property (iii) is easily seen to be equivalent to (i) in the one-dimensional setting. While it is – a priori – not clear how to extend the notion (i) of co-monotonicity to the vector-valued setting, the notion (iii) of strong coherence extends to the vector-valued case on an obvious way. This is the reason why Ekeland-Galichon-Henry (2009) introduced this notion (in the vector valued setting).

Here is the second theorem which we state for the one-dimensional case, and later extend to the $d$-dimensional one.
Theorem B (Kusuoka, 2001)

Let \( \rho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \) be a law invariant convex risk measure. Then there is a function \( \nu : [0, 1] \times \mathcal{P} \to [0, \infty] \) such that

\[
\rho(X) = \max_{(\alpha, F) \in [0, 1] \times \mathcal{P}} \left\{ \alpha \rho_F(X) + (1 - \alpha) \rho^\infty(X) - \nu(\alpha, F) \right\}.
\]
A convex risk measure on $\mathbb{R}^d$ is a function $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \rightarrow \mathbb{R}$ s.t.

(i) (normalization) $\varrho(0) = 0$,

(ii) (monotonicity) $X \geq Y \Rightarrow \varrho(X) \leq \varrho(Y)$,

(iii) (cash invariance) $\varrho(X + m1) = \varrho(X) - m$, for $m \in \mathbb{R}$,

(iv) (convexity) $\varrho(\alpha X + (1 - \alpha) Y) \leq \alpha \varrho(X) + (1 - \alpha) \varrho(Y)$, for $0 < \alpha < 1$.

We call $\varrho$ coherent if, in addition,

(v) (positive homogeneity) $\varrho(\lambda X) = \lambda \varrho(X)$, for $\lambda \geq 0$. 

W. Schachermayer joint work in progress with I. Ekeland

Law invariant convex risk measures on $\mathbb{R}^d$
On the standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\) we now denote by \(\mathcal{P}\) the subset of \(L^1_+(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)\) of normalized functions \(F\) taking their values in \(\mathbb{R}_+^d\),

\[
\mathcal{P} = \left\{ F = (F_i)_{i=1}^d \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) : F_i \geq 0, \mathbb{E} \left[ \sum_{i=1}^d F_i \right] = 1 \right\}.
\]
Definition (Rüschendorf)

For given $F \in \mathcal{P}$ we define $\varrho_F$, the *maximal correlation risk measure in the direction* $F$, by

$$
\varrho_F(X) = \sup_{\tilde{X} \sim X} \mathbb{E} \left[ - (\tilde{X}, F) \right] = \sup_{\tilde{F} \sim F} \mathbb{E} \left[ - (X, \tilde{F}) \right].
$$

Note that $\varrho_F$ only depends on the law of $F$. 
Proposition (Rüschendorf)

A coherent (resp. convex) risk measure \( \varrho : L^\infty_d \rightarrow \mathbb{R} \) is *law invariant* if and only if it can be represented as

\[
\varrho(X) = \sup_{F \in C} \varrho_F(X)
\]

resp. \( \varrho(X) = \sup_{F \in C} \{ \varrho_F(x) - \nu(F) \} \)

where \( C \) is a subset of \( \mathcal{P} \) and \( \nu : C \rightarrow \mathbb{R}_+ \) a non-negative function defined on \( C \).
Apart from the risk measures $\varrho_F$, where $F \in \mathcal{P}$, a second type of risk measures will play a special role. It generalizes the maximal loss measure from the one- to the $d$-dimensional case.
Definition

For \( \xi \in S^d \), where

\[
S^d := \left\{ \xi \in \mathbb{R}^d : \xi_i \geq 0, \sum_{i=1}^{d} \xi_i = 1 \right\},
\]

we define the maximal loss measure in the direction \( \xi \) by

\[
\varphi^\infty_\xi (X) = \text{ess sup} \left\{ - \sum_{i=1}^{d} \xi_i X_i \right\}.
\]

More generally, for a probability measure \( \mu \) on \( S^d \) we may define

\[
\varphi^\infty_\mu (X) = \int_{S^d} \varphi^\infty_\xi (X) d\mu(\xi).
\]
### Theorem (Ekeland, Galichon, Henry, 2009)

Assume that $\varrho$ is a convex, law invariant risk measure on $\mathbb{R}^d$ which extends continuously from $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ to $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, for some $p < \infty$. Then there is a function $\nu : \mathcal{P} \to [0, \infty]$ such that

$$
\varrho(X) = \max_{F \in \mathcal{P}} \{ \varrho_F(X) - \nu(F) \}
$$

### Theorem (Ekeland, S., 2010)

Assume that $\varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ is a convex, law invariant risk measure on $\mathbb{R}^d$. Then there is a function $\nu : [0, 1] \times \mathcal{P} \times \mathcal{M}_+(S^d) \to [0, \infty]$ such that

$$
\varrho(X) = \max_{(\alpha, F, \mu) \in [0,1] \times \mathcal{P} \times \mathcal{M}_+(S^d)} \left\{ \alpha \varrho_F(X) + (1 - \alpha) \varrho_\infty^\mu(X) - \nu(\alpha, F, \mu) \right\}
$$

The law invariant risk measure $\varrho$ is coherent if and only if $\nu$ can be chosen to take only values in $\{0, \infty\}$. 

W. Schachermayer joint work in progress with I. Ekeland

Law invariant convex risk measures on $\mathbb{R}^d$
Definition (Ekeland, Galichon, Henry, 2009)

A convex, law invariant risk measure \( \varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R} \) is **strongly coherent** if

\[
\varrho(X) + \varrho(Y) = \sup \left\{ \varrho(\tilde{X} + \tilde{Y}) : \tilde{X} \sim X, \tilde{Y} \sim Y \right\}.
\]
Theorem (Ekeland, Galichon, Henry, 2009)

Let \( \varrho \) be a convex, law invariant risk measure which extends continuously from \( L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \) to \( L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \), for some \( p < \infty \). Then \( \varrho \) is strongly coherent if and only if there is some \( F \in \mathcal{P} \) such that

\[
\varrho(X) = \alpha \varrho_F(X).
\]

Theorem (Ekeland, S., 2010)

A convex, law invariant risk measure \( \varrho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \rightarrow \mathbb{R} \) is strongly coherent if and only if there is some \( F \in \mathcal{P}, \mu \in \mathcal{M}_+(S^d) \) and \( \alpha \in [0, 1] \) such that

\[
\varrho(X) = \alpha \varrho_F(X) + (1 - \alpha) \varrho^\infty_\mu(X).
\]