Arbitrage in Market Models with a Stochastic Number of Assets

Winslow Strong

Department of Statistics and Applied Probability,
University of California Santa Barbara /
Visitor to Laboratoire de Probabilités et Modèles Aléatoires, Paris 6-7.

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Outline

1. Market Models with a Stochastic Number of Assets
   - Motivation
   - Piecewise Semimartingales
2. Arbitrage
   - Fundamental Theorems of Asset Pricing
   - Functionally Generated Relative Arbitrage

Outline

Motivation

- Usually, stock prices are modeled as $\mathbb{R}^n$-valued stochastic processes.
- Why allow the number of assets to be stochastic?
- Realism. Companies enter, leave, merge and split in real equity markets.
- The Market Portfolio is of central importance in modern portfolio theory of economics, and stochastic portfolio theory of continuous time finance.
- Question: Does a stochastic number of assets qualitatively change characterizations of arbitrage compared to constant-number-of-asset markets?

Outline

Regularity Assumptions

- The stock process X is progressive, has paths with left and right limits, and takes values in $\mathbb{U} := \cup_{n=1}^{\infty} \mathbb{R}^n$.
- The dimensional process $N := \dim X$ has paths that are left-continuous and piecewise constant in time.

$$\begin{aligned} \tau_0 &:= 0, \\ \tau_k &:= \inf\{t > \tau_{k-1} \mid X_t^+ \neq X_t\}, \quad k \in \mathbb{N} \end{aligned}$$

- Then N is constant in time on each $(\tau_{k-1}, \tau_k]$.
- A new piece begins at each right-discontinuity.
- Assume $\tau_k \nearrow \infty$, a.s.

Dissection

- Introduce the additive identity element $\odot$, so that $x + \odot = \odot + x = x$, $\forall x \in \mathbb{U} \cup \{\odot\}$.
- For stochastic process Y and $A \subseteq \mathbb{R}_+ \times \Omega$, define the operation $\star$:

$$(Y \star \mathbf{1}_A)_t(\omega) = (\mathbf{1}_A \star Y)_t(\omega) := \begin{cases} Y_t(\omega) & \text{for } (t, \omega) \in A \\ \odot & \text{otherwise} \end{cases}.$$

- Dissection: Chop up X into $\mathbb{R}^n$-valued processes on each $(\tau_{k-1}, \tau_k]$. For each $(k, n) \in \mathbb{N}^2$, define

$$\begin{aligned} 0^{(n)} &:= (0, \dots, 0) \in \mathbb{R}^n, \\ A_{k,n} &:= (\tau_{k-1}, \infty) \cap [0, \infty) \times \{\tau_{k-1} < \infty, N^+_{\tau_{k-1}} = n\} \subseteq \mathbb{R}_+ \times \Omega, \\ X^{k,n} &:= (X^{\tau_k} - X^+_{\tau_{k-1}}) \star \mathbf{1}_{A_{k,n}} + 0^{(n)} \star \mathbf{1}_{A^c_{k,n}}. \end{aligned}$$

- Then $X^{k,n}$ is an $\mathbb{R}^n$-valued process, $\forall (k, n) \in \mathbb{N}^2$.

Extension of Stochastic Integration

Definition

X is a *$\mathbb{U}$-valued piecewise semimartingale* if $X^{k,n}$ is an $\mathbb{R}^n$-valued semimartingale for each $(k,n) \in \mathbb{N}^2$.

- Assume H is predictable and satisfies $\dim H = N$. Dissect:

$$B_{k,n} := (\tau_{k-1}, \tau_k] \cap [0,\infty) \times \{\tau_{k-1} < \infty, N^+_{\tau_{k-1}} = n\},$$

$$H^{k,n} := H \star \mathbf{1}_{B_{k,n}} + 0^{(n)} \star \mathbf{1}_{B^c_{k,n}}.$$

- If each $H^{k,n}$ is $X^{k,n}$-integrable, in the sense of $\mathbb{R}^n$-valued semimartingale integration, then we say $H \in \mathscr{L}(X)$, and

$$H \cdot X := \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} H^{k,n} \cdot X^{k,n}.$$

- Stochastic integral $H \cdot X$ extends $\mathbb{R}^n$-stochastic integration.
- Retains that $\cdot X$ is a continuous linear operator on the appropriate generalization of simple, predictable processes.

$\mathbb{U}$-valued Piecewise Martingales

- A characterization of martingality of X by conditional expectation is not useful or appropriate.
- Instead, characterize via martingality of $H \cdot X$.

Definition

X is a *$\mathbb{U}$-valued piecewise martingale* if $H \cdot X$ is an $\mathbb{R}$-valued martingale $\forall H$: simple, predictable, $\dim H = \dim X$, $|H|$ bounded. X is a *$\mathbb{U}$-valued piecewise local martingale* if X is locally a *$\mathbb{U}$-valued piecewise martingale*. X is *$\mathbb{U}$-valued piecewise σ-martingale* if $H \cdot X$ is a σ-martingale for all $H \in \mathscr{L}(X)$.

- All of these are necessary and sufficient when X is an $\mathbb{R}^n$-valued semimartingale, so are proper extensions.

Outline

The Market Model

- *Self financing:*

$$V_t = V_0 + (H \cdot X)_t, \quad t \geq 0.$$

- The market changes configuration at each τ_k, but $H \cdot X$ is always right-continuous by its definition.
- The implicit assumption is that portfolio values are unaffected by the changes in market configuration at τ_k^+.
- The model *can* handle events such as a stock jumping to bankruptcy. The jump to 0 occurs via a left-discontinuity, affecting $H \cdot X$, and then the company may be removed from the market via a right-discontinuity: $N_{\tau_k^+} = N_{\tau_k} - 1$.
- Trading process $H \in \mathscr{L}(X)$ is *admissible* if there exists $c \in \mathbb{R}$:

$$(H \cdot X)_t \geq -c, \quad \forall t \geq 0.$$

Extension of NFLVR Equivalence

Theorem (Mémin extension)

$\{H \cdot X \mid H \in \mathscr{L}(X)\}$ is closed in the semimartingale topology.

Proof.

Localize. Each $\{H^{k,n} \cdot X^{k,n} \mid H^{k,n} \in \mathscr{L}(X^{k,n})\}$ is closed by Mémin. $\square$

Theorem (FTAP extension)

NFLVR $\iff$ existence of an equivalent measure under which $H \cdot X$ is a supermartingale $\forall H$ admissible.

Proof.

Immediate from the Mémin extension via [Kabanov(1997)]. $\square$

I have not proved a σ-martingale characterization for X yet.

Corollaries

Corollary

If X is an $\mathbb{R}^n$-valued semimartingale, and $|X|$ is bounded, then NFLVR $\iff$ existence of an equivalent martingale measure for X.

But since X may have right-discontinuities not passed on to $H \cdot X$,

Fact

[$|X|$ bounded $\cap$ NFLVR] $\nRightarrow$ existence of an equivalent martingale measure for X.

Corollary

If $|X|$ is locally bounded, then NFLVR $\iff$ existence of an equivalent local martingale measure for X.

NA_1 and Equivalent Local Martingale Deflator

Definition

An *arbitrage of the first kind* for X on horizon α, a stopping time, is an $\mathscr{F}_\alpha$-measurable random variable ψ such that $P[\psi \geq 0] = 1$, $P[\psi > 0] > 0$ and, for each $v > 0$, there exists H such that $v + H \cdot X \geq 0$, and $v + (H \cdot X)_\alpha \geq \psi$. If there does not exist any arbitrage of the first kind, then we say NA_1 holds.

NA_1 is weaker than NFLVR. [Kardaras(2009)] proves the FTAP for NA_1, relating it to ELMD.

Definition

An *equivalent local martingale deflator* (ELMD) for X is a strictly positive $\mathbb{R}$-valued local martingale Z, such that for each H admissible, $Z(H \cdot X)$ is a local martingale.

Extension of the NA_1-FTAP

Theorem [Kardaras(2009)]

Let α be a stopping time and X an $\mathbb{R}^n$-valued semimartingale. NA_1 holds for X on horizon α if and only if there exists an ELMD for X on horizon α.

The next theorem is my extension.

Theorem

Let α be a stopping time and X an $\mathbb{U}$-valued piecewise semimartingale. NA_1 holds for X on horizon α if and only if it holds for each $X^{k,n}$, $(k,n) \in \mathbb{N}^2$, on horizon α if and only if there exists an ELMD for X on horizon α.

- This FTAP is much easier to check in practice than the Delbaen-Schachermayer FTAP.
- For many applications (portfolio optimization, hedging) it provides sufficient market regularity and greater flexibility.

Outline

Background on Stochastic Portfolio Theory (SPT) in $\mathbb{R}^n$

- Begun with [Fernholz and Shay(1982)] and continued by R. Fernholz in the late 1990s, leading to the monograph [Fernholz(2002)].
- Recent work by: Karatzas, Kardaras, D. Fernholz, Pal, Ichiba, many others.
- Motivated by: The robust empirical outperformance of constant weight portfolios compared to their passive counterparts.
- Goals: Understand what type of models reproduce this, and if there are fundamental properties of real markets that explain it. *Let the data guide the theory!*
- Itô process model for the n stocks:

$$dX_t^i = X_t^i \left[b_t^i dt + \sum_{\nu=1}^{d} \sigma_t^{i\nu} dW_t^{\nu} \right], \quad 1 \leq i \leq n.$$

- Assume also *uniform ellipticity* of the covariance:

$$\exists \varepsilon > 0 : \qquad \xi \sigma_t \sigma_t^\top \xi \geq \varepsilon |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall t \geq 0.$$

Diversity and Arbitrage

- Empirical observation: modern financial markets do not permit any company to approach the size of the entire market.
- Mathematical formulation: WLOG let X^i be the total capitalization (#shares $\times$price per share) of stock i.
- The *market portfolio* μ plays an important role:

$$\mu^i := X^i / \sum_{j=1}^{n} X^j, \quad \mu_t(\omega) \in \Delta := \{u \mid u^i \geq 0, \forall i, \sum u^i = 1\}, \quad \forall t, \omega.$$

Diversity: There exists some $\delta \in (0,1)$ such that

$$\mu_t^{(1)} := \max_{1 \leq i \leq n} \mu_t^i \leq 1 - \delta, \quad \forall t \geq 0.$$

- Consequences: [uniform ellipticity $\cap$ diversity] $\Rightarrow$ there exist arbitrages relative to the market portfolio that require no knowledge of σ or b to construct. These portfolios are *functionally generated* from the market portfolio.

Functionally Generated Portfolios

- Let V^π be the wealth process of a portfolio π, where π^i is the fraction of V^π invested in X^i.

$$\frac{dV_t^\pi}{V_t^\pi} = \sum_{i=1}^n \pi_t^i \frac{dX_t^i}{X_t^i}$$

- A *portfolio generating function* is a function $G : U \subseteq \Delta \to (0, \infty)$ such that $G \in C^2$, and additional mild regularity.
- Solve for the (unique) π such that

$$d \log V_t^\pi = d \log V_t^\mu + d \log G(\mu_t) + (?) dt.$$

- This is analogous to deriving the hedging portfolio for contingent claim $V^\mu G(\mu)$.

Master Formula

- This π corresponding to G is called the portfolio generated by G. It obeys the *master formula*

$$\log\left(\frac{V_T^\pi}{V_T^\mu}\right) = \log G(\mu_T) - \log G(\mu_0) + \int_0^T \mathfrak{g}_t\, dt. \qquad (1)$$

- If G is concave and symmetric, then $\mathfrak{g} \geq 0$, and π is long only. Such a π buys a little bit of a stock each time it falls in price relative to the others, and sells a little each time it rises.
- The values for $\mathfrak{g}$ and π:

$$\mathfrak{g}_t = \frac{-1}{2G(\mu_t)} \sum_{i,j} \left[\frac{\partial^2 G(\mu_t)}{\partial \mu_i \partial \mu_j}\right] \left[\frac{d}{dt}\left\langle \mu_t^i, \mu_t^j \right\rangle_t\right],$$

$$\pi_t^i = \mu_t^i \left(\frac{\partial}{\partial \mu_i} G(\mu_t) + 1 - \sum_{j=1}^n \mu_t^j \frac{\partial}{\partial \mu_j} G(\mu_t)\right), \quad 1 \leq i \leq n.$$

- Key fact: The drift b of X does not appear in (1)!

Relative Arbitrage

- Some nice choices for G:

$$
\begin{aligned}
G_p(u) &:= \Big(\sum (u^i)^p\Big)^{1/p}, \quad p \in (0,1),\\
G_{\mathrm{e}}(u) &:= -\sum u^i \log u^i,\\
G_\psi(u) &:= (u^1)^{\psi^1} \times \ldots \times (u^n)^{\psi^n} \Rightarrow \pi = \psi, \text{ for } \psi \in \Delta,\\
G^c(u) &:= c + G(u), \quad c \in (0,\infty).
\end{aligned}
$$

- When diversity and uniform ellipticity hold, then G_p, G_{e}^c, G_ψ^c all satisfy: $\log G(\mu) \geq -\kappa$, $\kappa \in (0,\infty)$, and $\mathfrak{g} \geq \gamma \in (0,\infty)$.
- This implies that $V_T^\pi > V_T^\mu$ for all $T > T^* := \frac{\kappa + \log G(\mu_0)}{\gamma}$, so π *is an arbitrage relative to* μ *on horizon* T.

- To study functionally generated arbitrage when the number of assets is stochastic, it is appropriate to adopt a $\mathbb{U}$-valued piecewise Itô process model.
- Let X be a $\mathbb{U}$-valued piecewise Itô process

$$dX_t = X_t[b_t dt + \sigma_t dW_t], \quad \text{on each } (\tau_{k-1}, \tau_k].$$

- The *market portfolio* is

$$\mu_t^i = \mu_t^i(X) := \frac{X_t^i}{\sum_{j=1}^{N_t} X_t^j}, \quad 1 \leq i \leq N_t,\ t \geq 0.$$

Functionally Generated Arbitrage

- When X is $\mathbb{R}^n$-valued, recall the master formula:

$$\log\left(\frac{V_T^\pi}{V_T^\mu}\right) = \log G(\mu_T) - \log G(\mu_0) + \int_0^T \mathfrak{g}(t)dt.$$

- This was derived by an application of Itô's formula, and choosing π to eliminate the stochastic integral.
- If X is $\mathbb{U}$-valued Itô, then Itô's formula holds on each $(\tau_{k-1}, \tau_k]$, so the master formula generalizes to

$$\log\left(\frac{V_T^\pi}{V_T^\mu}\right) = \sum_{k=1}^{K_T}\left(\log G(\mu_{\tau_k}) - \log G(\mu_{\tau_{k-1}^+})\right)$$
$$+ \log G(\mu_T) - \log G(\mu_{\tau_{K_T}^+}) + \int_0^T \mathfrak{g}(t)dt,$$

$$K_T := \sum_{k=1}^{\infty} \mathbf{1}_{T>\tau_k}.$$

Example: Diverse Market that Grows at Times $\perp W$

- Suppose that X is strong Markov at the τ_k, which themselves are independent of W.
- Assume: $P(K_T > k) > 0$, $\forall k \in \mathbb{N}$.
- Let X be diverse: for some $n_0 \geq 2$, $\delta \in (0, \frac{n_0 - 1}{n_0})$ let

$$U^n := \{x \in (0,\infty)^n \mid \mu^{(1)}(x) < 1 - \delta\}.$$

- Suppose that μ *communicates* on each $\mu(U^n)$, meaning roughly that it has strictly positive probability of reaching any neighborhood in $\mu(U^n)$ from any point in arbitrarily small time, $\forall n \in \mathbb{N}$. See [Strong(2010)] for a precise statement.
- Let the covariance satisfy:

$$a_{\min} |\xi|^2 \leq \xi' a_t \xi \leq a_{\max} |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall n \in \mathbb{N}, \forall t \geq 0.$$

- Let the market grow from $N_0^+ = n_0 \in \mathbb{N}$ companies by new companies entering the market, one at each τ_k, $k \in \mathbb{N}$, so that

$$\mu(X_{\tau_k^+}) = ((1-\mu^{\text{new}})(\mu_{\tau_k}), \mu^{\text{new}}),$$

  where the relative size μ^{new} of the new company has support that in a subset of $[\varepsilon_l, \varepsilon_u]$ for any $0 < \varepsilon_l < \varepsilon_u < 1-\delta$.
- Then the entropy-weighted G_e^c and diversity-p G_p generating functions satisfy $G(\mu(X_{\tau_k^+})) > \inf_{x \in U^n} G \restriction_{U^n} (\mu(x)) + \varepsilon$, a.s. on $\{N_{\tau_k^+} = n\}$, for all $(k,n) \in \mathbb{N}^2$.
- Furthermore $P(\log G(\mu_{\tau_k}) - \log G(\mu_{\tau_{k-1}^+}) < -\frac{\varepsilon}{2} \mid \mathscr{F}_{\tau_{k-1}}) > 0$ a.s.
- This means that there is always a chance of losing at least $\frac{\varepsilon}{2}$ in $\log G(\mu)$ on $(\tau_{k-1}, \tau_k]$. Therefore

$$P\left(\sum_{k=1}^{K_T+1} \log\left(\frac{G(\mu_{\tau_k \wedge T})}{G(\mu_{\tau_{k-1}^+ \wedge T})}\right) < -\kappa\right) > 0, \quad \forall \kappa \in \mathbb{R}.$$

- $\mathfrak{g}$ is bounded from above uniformly in time. Therefore the master formula

$$\log\left(\frac{V_T^\pi}{V_T^\mu}\right) = \sum_{k=1}^{K_T}\left(\log G(\mu_{\tau_k}) - \log G(\mu_{\tau_{k-1}^+})\right) + \log G(\mu_T) - \log G(\mu_{\tau_{K_T}^+}) + \int_0^T \mathfrak{g}(t)dt,$$

$$K_T := \sum_{k=1}^{\infty} \mathbf{1}_{T>\tau_k}.$$

implies that relative losses are unbounded:

$$P\left(\frac{V_T^\pi}{V_T^\mu} < \frac{1}{\theta}\right) > 0, \quad \forall \theta \in \mathbb{R}.$$

- This market is diverse and uniformly elliptic, but does not admit straitforward functionally generated arbitrage.
- Due to the diversity, and covariance being bounded from above, the market has no ELMM, so admits FLVR.
- Open question: Does it admit any (non-straightforward) functionally generated arbitrage?

Summary

- Semimartingale stochastic integration may be extended to $\mathbb{U}$-valued piecewise semimartingale stochastic integration.
- The NFLVR equivalence to the existence of an equivalent pricing measure, and its related theorems extend.
- The NA_1 equivalence to existence of an equivalent local martingale deflator extends as well.
- Functionally generated portfolios are susceptible to the changes in configuration of the market.
- If K and N are bounded, they may work but take longer.
- If K is unbounded, then the portfolios typically fail to bound worst case relative performance.

Open Questions and Future Work

- Even though functionally generate portfolios typically fail to be arbitrages in $\mathbb{U}$-valued models, under what conditions do they satisfy weaker outperformance criteria (e.g. superior asymptotic growth)?
- Are there (non-straightforward) functionally generated arbitrages for the models in this talk?
- Explore the interaction between market growth, stability and diversity over time.

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