Arbitrage in Market Models with a Stochastic Number of Assets

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1. Market Models with a Stochastic Number of Assets
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Motivation

- Usually, stock prices are modeled as $\mathbb{R}^n$-valued stochastic processes.
- Why allow the number of assets to be stochastic?
- **Realism.** Companies enter, leave, merge and split in real equity markets.
- **The Market Portfolio** is of central importance in modern portfolio theory of economics, and stochastic portfolio theory of continuous time finance.
- **Question:** Does a stochastic number of assets qualitatively change characterizations of arbitrage compared to constant-number-of-asset markets?
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The stock process $X$ is progressive, has paths with left and right limits, and takes values in $\mathbb{U} := \bigcup_{n=1}^{\infty} \mathbb{R}^n$.

The dimensional process $N := \dim X$ has paths that are left-continuous and piecewise constant in time.

\[
\tau_0 := 0,
\]

\[
\tau_k := \inf\{t > \tau_{k-1} \mid X_t^+ \neq X_t\}, \quad k \in \mathbb{N}
\]

Then $N$ is constant in time on each $(\tau_{k-1}, \tau_k]$.

A new piece begins at each right-discontinuity.

Assume $\tau_k \uparrow \infty$, a.s.
Dissection

- Introduce the additive identity element ⊙, so that \( x + ⊙ = ⊙ + x = x, \forall x \in \mathbb{U} \cup \{⊙\} \).

- For stochastic process \( Y \) and \( A \subseteq \mathbb{R}_+ \times \Omega \), define the operation \( \star \):

\[
(Y \star 1_A)_t(\omega) = (1_A \star Y)_t(\omega) := \begin{cases} 
Y_t(\omega) & \text{for } (t, \omega) \in A \\
⊙ & \text{otherwise}
\end{cases}
\]

- **Dissection**: Chop up \( X \) into \( \mathbb{R}^n \)-valued processes on each \((\tau_{k-1}, \tau_k] \). For each \((k, n) \in \mathbb{N}^2 \), define

\[
0^{(n)} := (0, \ldots, 0) \in \mathbb{R}^n, \\
A_{k, n} := (\tau_{k-1}, \infty) \cap [0, \infty) \times \{\tau_{k-1} < \infty, N_{\tau_{k-1}} = n\} \subseteq \mathbb{R}_+ \times \Omega, \\
X_{k, n} := (X_{\tau_k} - X_{\tau_{k-1}}^+) \star 1_{A_{k, n}} + 0^{(n)} \star 1_{A_{k, n}^c}.
\]

- Then \( X_{k, n} \) is an \( \mathbb{R}^n \)-valued process, \( \forall (k, n) \in \mathbb{N}^2 \).
Extension of Stochastic Integration

**Definition**

*X* is a **U-valued piecewise semimartingale** if *X*<sub>k,n</sub> is an \( \mathbb{R}^n \)-valued semimartingale for each \((k, n) \in \mathbb{N}^2\).

- Assume \( H \) is predictable and satisfies \( \dim H = N \). Dissect:

  \[
  B_{k,n} := (\tau_{k-1}, \tau_k] \cap [0, \infty) \times \{ \tau_{k-1} < \infty, N^+_{\tau_{k-1}} = n \},
  \]

  \[
  H^{k,n} := H \ast 1_{B_{k,n}} + 0^{(n)} \ast 1_{B_{k,n}^c}.
  \]

- If each \( H^{k,n} \) is \( X^{k,n} \)-integrable, in the sense of \( \mathbb{R}^n \)-valued semimartingale integration, then we say \( H \in \mathcal{L}(X) \), and

  \[
  H \cdot X := \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} H^{k,n} \cdot X^{k,n}.
  \]

- Stochastic integral \( H \cdot X \) extends \( \mathbb{R}^n \)-stochastic integration.
- Retains that \( \cdot X \) is a continuous linear operator on the appropriate generalization of simple, predictable processes.
A characterization of martingality of $X$ by conditional expectation is not useful or appropriate.

Instead, characterize via martingality of $H \cdot X$.

**Definition**

$X$ is a \textbf{$U$-valued piecewise martingale} if $H \cdot X$ is an $\mathbb{R}$-valued martingale $\forall H: \text{simple, predictable, dim } H = \text{dim } X$, $|H|$ bounded.

$X$ is a \textbf{$U$-valued piecewise local martingale} if $X$ is locally a \textbf{$U$-valued piecewise martingale}. $X$ is \textbf{$U$-valued piecewise $\sigma$-martingale} if $H \cdot X$ is a $\sigma$-martingale for all $H \in \mathcal{L}(X)$.

All of these are necessary and sufficient when $X$ is an $\mathbb{R}^n$-valued semimartingale, so are proper extensions.
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**Self financing:**

\[ V_t = V_0 + (H \cdot X)_t, \quad t \geq 0. \]

- The market changes configuration at each \( \tau_k \), but \( H \cdot X \) is always right-continuous by its definition.
- The implicit assumption is that portfolio values are unaffected by the changes in market configuration at \( \tau_k^+ \).
- The model *can* handle events such as a stock jumping to bankruptcy. The jump to 0 occurs via a left-discontinuity, affecting \( H \cdot X \), and then the company may be removed from the market via a right-discontinuity: \( N_{\tau_k^+} = N_{\tau_k} - 1 \).
- Trading process \( H \in \mathcal{L}(X) \) is *admissible* if there exists \( c \in \mathbb{R} \):
  \[ (H \cdot X)_t \geq -c, \quad \forall t \geq 0. \]
Extension of NFLVR Equivalence

**Theorem (Mémin extension)**

\[ \{ H \cdot X \mid H \in \mathcal{L}(X) \} \] is closed in the semimartingale topology.

**Proof.**

Localize. Each \( \{ H^k \cdot X^k, n \mid H^k, n \in \mathcal{L}(X^k, n) \} \) is closed by Mémin.

**Theorem (FTAP extension)**

NFLVR \( \iff \) existence of an equivalent measure under which \( H \cdot X \) is a supermartingale \( \forall H \) admissible.

**Proof.**

Immediate from the Mémin extension via [Kabanov(1997)].

I have not proved a \( \sigma \)-martingale characterization for \( X \) yet.
Corollary

If $X$ is an $\mathbb{R}^n$-valued semimartingale, and $|X|$ is bounded, then $\text{NFLVR} \iff \text{existence of an equivalent martingale measure for } X$.

But since $X$ may have right-discontinuities not passed on to $H \cdot X$,

Fact

$[|X|$ bounded $\cap$ NFLVR] $\nRightarrow$ existence of an equivalent martingale measure for $X$.

Corollary

If $|X|$ is locally bounded, then $\text{NFLVR} \iff \text{existence of an equivalent local martingale measure for } X$. 
Definition

An *arbitrage of the first kind* for $X$ on horizon $\alpha$, a stopping time, is an $\mathcal{F}_\alpha$-measurable random variable $\psi$ such that $P[\psi \geq 0] = 1$, $P[\psi > 0] > 0$ and, for each $\nu > 0$, there exists $H$ such that $\nu + H \cdot X \geq 0$, and $\nu + (H \cdot X)_\alpha \geq \psi$. If there does not exist any arbitrage of the first kind, then we say $\text{NA}_1$ holds.

$\text{NA}_1$ is weaker than NFLVR. [Kardaras(2009)] proves the FTAP for $\text{NA}_1$, relating it to ELMD.

Definition

An *equivalent local martingale deflator* (ELMD) for $X$ is a strictly positive $\mathbb{R}$-valued local martingale $Z$, such that for each $H$ admissible, $Z(H \cdot X)$ is a local martingale.
Theorem [Kardaras(2009)]

Let $\alpha$ be a stopping time and $X$ an $\mathbb{R}^n$-valued semimartingale. $\text{NA}_1$ holds for $X$ on horizon $\alpha$ if and only if there exists an ELMD for $X$ on horizon $\alpha$.

The next theorem is my extension.

Theorem

Let $\alpha$ be a stopping time and $X$ an $\mathbb{U}$-valued piecewise semimartingale. $\text{NA}_1$ holds for $X$ on horizon $\alpha$ if and only if it holds for each $X^{k,n}$, $(k,n) \in \mathbb{N}^2$, on horizon $\alpha$ if and only if there exists an ELMD for $X$ on horizon $\alpha$.

- This FTAP is much easier to check in practice than the Delbaen-Schachermayer FTAP.
- For many applications (portfolio optimization, hedging) it provides sufficient market regularity and greater flexibility.
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Background on Stochastic Portfolio Theory (SPT) in $\mathbb{R}^n$

- Begun with [Fernholz and Shay(1982)] and continued by R. Fernholz in the late 1990s, leading to the monograph [Fernholz(2002)].
- Recent work by: Karatzas, Kardaras, D. Fernholz, Pal, Ichiba, many others.
- Motivated by: The robust empirical outperformance of constant weight portfolios compared to their passive counterparts.
- Goals: Understand what type of models reproduce this, and if there are fundamental properties of real markets that explain it. *Let the data guide the theory!*

Itô process model for the $n$ stocks:

$$dX^i_t = X^i_t \left[ b^i_t dt + \sum_{\nu=1}^{d} \sigma^{i\nu}_t dW^\nu_t \right], \quad 1 \leq i \leq n.$$ 

Assume also *uniform ellipticity* of the covariance:

$$\exists \varepsilon > 0 : \quad \xi \sigma_t \sigma^\top_t \xi \geq \varepsilon |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall t \geq 0.$$
Diversity and Arbitrage

- **Empirical observation:** modern financial markets do not permit any company to approach the size of the entire market.

- **Mathematical formulation:** WLOG let $X^i$ be the total capitalization (\#shares $\times$ price per share) of stock $i$.

- The *market portfolio* $\mu$ plays an important role:

$$
\mu^i := X^i / \sum_{j=1}^{n} X^j, \quad \mu_t(\omega) \in \Delta := \{ u \mid u^i \geq 0, \forall i, \sum u^i = 1 \}, \quad \forall t, \omega.
$$

**Diversity:** There exists some $\delta \in (0,1)$ such that

$$
\mu_t^{(1)} := \max_{1 \leq i \leq n} \mu^i_t \leq 1 - \delta, \quad \forall t \geq 0.
$$

- **Consequences:** [uniform ellipticity $\cap$ diversity] $\Rightarrow$ there exist arbitrage relative to the market portfolio that require no knowledge of $\sigma$ or $b$ to construct. These portfolios are *functionally generated* from the market portfolio.
Let $V^{\pi}$ be the wealth process of a portfolio $\pi$, where $\pi^i$ is the fraction of $V^{\pi}$ invested in $X^i$.

$$\frac{dV^\pi_t}{V^\pi_t} = \sum_{i=1}^n \pi^i_t \frac{dX^i_t}{X^i_t}$$

A portfolio generating function is a function $G : U \subseteq \Delta \rightarrow (0, \infty)$ such that $G \in C^2$, and additional mild regularity.

Solve for the (unique) $\pi$ such that

$$d \log V^\pi_t = d \log V^\mu_t + d \log G(\mu_t) + (?) dt.$$ 

This is analogous to deriving the hedging portfolio for contingent claim $V^\mu G(\mu)$. 
This π corresponding to G is called the portfolio generated by G. It obeys the *master formula*

$$\log \left( \frac{V_\pi^T}{V_\mu^T} \right) = \log G(\mu_T) - \log G(\mu_0) + \int_0^T g_t \, dt. \quad (1)$$

If G is concave and symmetric, then $g \geq 0$, and π is long only. Such a π buys a little bit of a stock each time it falls in price relative to the others, and sells a little each time it rises.

The values for $g$ and $\pi$:

$$g_t = \frac{-1}{2G(\mu_t)} \sum_{i,j} \left[ \frac{\partial^2 G(\mu_t)}{\partial \mu_i \partial \mu_j} \right] \left[ \frac{d}{dt} \langle \mu_t^i, \mu_t^j \rangle_t \right],$$

$$\pi_t^i = \mu_t^i \left( \frac{\partial}{\partial \mu_i} G(\mu_t) + 1 - \sum_{j=1}^n \mu_t^j \frac{\partial}{\partial \mu_j} G(\mu_t) \right), \quad 1 \leq i \leq n.$$

**Key fact:** The drift $b$ of $X$ does not appear in (1)!
Some nice choices for $G$:

$$G_p(u) := \left(\sum (u^i)^p\right)^{1/p}, \quad p \in (0,1),$$
$$G_e(u) := -\sum u^i \log u^i,$$
$$G_\psi(u) := (u^1)^{\psi^1} \times \ldots \times (u^n)^{\psi^n} \Rightarrow \pi = \psi, \text{ for } \psi \in \Delta,$$
$$G^c(u) := c + G(u), \quad c \in (0,\infty).$$

When diversity and uniform ellipticity hold, then $G_p$, $G^c$, $G_\psi$ all satisfy: $\log G(\mu) \geq -\kappa$, $\kappa \in (0,\infty)$, and $g \geq \gamma \in (0,\infty)$.

This implies that $V^\pi_T > V^\mu_T$ for all $T > T^* := \frac{\kappa + \log G(\mu_0)}{\gamma}$, so $\pi$ is an arbitrage relative to $\mu$ on horizon $T$. 
To study functionally generated arbitrage when the number of assets is stochastic, it is appropriate to adopt a $\mathbb{U}$-valued piecewise Itô process model.

Let $X$ be a $\mathbb{U}$-valued piecewise Itô process

$$dX_t = X_t[b_t dt + \sigma_t dW_t], \quad \text{on each } (\tau_{k-1}, \tau_k].$$

The \textit{market portfolio} is

$$\mu_t^i = \mu_t^i(X) := \frac{X_t^i}{\sum_{j=1}^{N_t} X_t^j}, \quad 1 \leq i \leq N_t, \ t \geq 0.$$
When $X$ is $\mathbb{R}^n$-valued, recall the master formula:

$$\log \left( \frac{V_{\pi T}}{V_{\mu T}} \right) = \log G(\mu_T) - \log G(\mu_0) + \int_0^T g(t) dt.$$ 

This was derived by an application of Itô’s formula, and choosing $\pi$ to eliminate the stochastic integral.

If $X$ is $\mathbb{U}$-valued Itô, then Itô’s formula holds on each $(\tau_{k-1}, \tau_k]$, so the master formula generalizes to

$$\log \left( \frac{V_{\pi T}}{V_{\mu T}} \right) = \sum_{k=1}^{K_T} \left( \log G(\mu_{\tau_k}) - \log G(\mu_{\tau_k^+}) \right)$$

$$+ \log G(\mu_T) - \log G(\mu_{\tau_{K_T}^+}) + \int_0^T g(t) dt,$$

$$K_T := \sum_{k=1}^{\infty} 1_{T > \tau_k}.$$
Suppose that $X$ is strong Markov at the $\tau_k$, which themselves are independent of $W$.

Assume: $P(K_T > k) > 0$, $\forall k \in \mathbb{N}$.

Let $X$ be diverse: for some $n_0 \geq 2$, $\delta \in (0, \frac{n_0-1}{n_0})$ let

$$U^n := \{ x \in (0, \infty)^n \mid \mu^{(1)}(x) < 1 - \delta \}.$$ 

Suppose that $\mu$ communicates on each $\mu(U^n)$, meaning roughly that it has strictly positive probability of reaching any neighborhood in $\mu(U^n)$ from any point in arbitrarily small time, $\forall n \in \mathbb{N}$. See [Strong(2010)] for a precise statement.

Let the covariance satisfy:

$$a_{\min} |\xi|^2 \leq \xi^\prime a_t \xi \leq a_{\max} |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall n \in \mathbb{N}, \forall t \geq 0.$$
Let the market grow from \( N_0^+ = n_0 \in \mathbb{N} \) companies by new companies entering the market, one at each \( \tau_k, k \in \mathbb{N} \), so that

\[
\mu(X_{\tau_k^+}) = ((1 - \mu^{\text{new}})(\mu_{\tau_k}), \mu^{\text{new}}),
\]

where the relative size \( \mu^{\text{new}} \) of the new company has support that in a subset of \([\varepsilon_l, \varepsilon_u]\) for any \( 0 < \varepsilon_l < \varepsilon_u < 1 - \delta \).

Then the entropy-weighted \( G_\xi \) and diversity-\( p \) \( G_p \) generating functions satisfy \( G(\mu(X_{\tau_k^+})) > \inf_{x \in U^n} G \upharpoonright U^n (\mu(x)) + \varepsilon \), a.s. on \( \{N_{\tau_k^+} = n\} \), for all \((k, n) \in \mathbb{N}^2\).

Furthermore \( P(\log G(\mu_{\tau_k}) - \log G(\mu_{\tau_{k-1}^+}) < -\frac{\varepsilon}{2} \mid \mathcal{F}_{\tau_{k-1}}) > 0 \) a.s.

This means that there is always a chance of losing at least \( \frac{\varepsilon}{2} \) in \( \log G(\mu) \) on \((\tau_{k-1}, \tau_k]\). Therefore

\[
P \left( \sum_{k=1}^{K_T+1} \log \left( \frac{G(\mu_{\tau_k} \wedge T)}{G(\mu_{\tau_{k-1}^+} \wedge T)} \right) < -\kappa \right) > 0, \quad \forall \kappa \in \mathbb{R}.
\]
• $g$ is bounded from above uniformly in time. Therefore the master formula

$$\log \left( \frac{V_\pi^T}{V_\mu^T} \right) = \sum_{k=1}^{K_T} \left( \log G(\mu_{\tau_k}) - \log G(\mu_{\tau_{k-1}^+}) \right)$$

$$+ \log G(\mu_T) - \log G(\mu_{\tau_{K_T}^+}) + \int_0^T g(t) dt,$$

$$K_T := \sum_{k=1}^\infty \mathbf{1}_{T > \tau_k}.$$ 

implies that relative losses are unbounded:

$$P \left( \frac{V_\pi^T}{V_\mu^T} < \frac{1}{\theta} \right) > 0, \quad \forall \theta \in \mathbb{R}.$$ 

• This market is diverse and uniformly elliptic, but does not admit straightforward functionally generated arbitrage.

• Due to the diversity, and covariance being bounded from above, the market has no ELMM, so admits FLVR.

• **Open question:** Does it admit any (non-straightforward) functionally generated arbitrage?
Summary

- Semimartingale stochastic integration may be extended to $\mathbb{U}$-valued piecewise semimartingale stochastic integration.
- The NFLVR equivalence to the existence of an equivalent pricing measure, and its related theorems extend.
- The NA$_1$ equivalence to existence of an equivalent local martingale deflator extends as well.
- Functionally generated portfolios are susceptible to the changes in configuration of the market.
- If $K$ and $N$ are bounded, they may work but take longer.
- If $K$ is unbounded, then the portfolios typically fail to bound worst case relative performance.
Open Questions and Future Work

- Even though functionally generate portfolios typically fail to be arbitrages in \( \mathbb{U} \)-valued models, under what conditions do they satisfy weaker outperformance criteria (e.g. superior asymptotic growth)?
- Are there (non-straightforward) functionally generated arbitrages for the models in this talk?
- Explore the interaction between market growth, stability and diversity over time.
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