Rating based Lévy Libor model

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Post-crisis interest rate markets

- a number of anomalies has appeared in the interest rate markets after the credit crisis at the end of 2007
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Introduction

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Modeling of credit risk in market models

- in mathematical finance defaultable interest rate models are often obtained by adding the defaultable term structure to the existing default-free term structure models in an appropriate way
- various defaultable extensions of the Heath–Jarrow–Morton (HJM) modeling methodology are found in the literature, whereas credit risk in Libor market models is far less studied (only papers by Lotz and Schlögl (2000), Schönbucher (2000), Eberlein, Kluge, and Schönbucher (2006))
- none of the existing defaultable Libor market models incorporates ratings and credit migration

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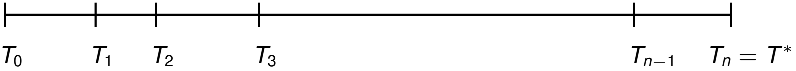

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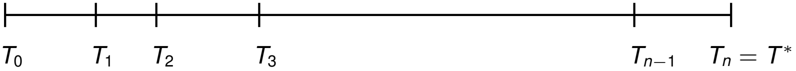

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Forward Libor rate at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$

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Defaultable zero coupon bonds with credit ratings: $B_C(\cdot, T_1), \ldots, B_C(\cdot, T_n)$

Defaultable forward Libor rate at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$

$$L_C(t, T_k) = \frac{1}{\delta_k}\left(\frac{B_C(t, T_k)}{B_C(t, T_{k+1})} - 1\right)$$

Libor modeling

- modeling under forward martingale measures, i.e. risk-neutral measures that use zero-coupon bonds as numeraires
- on a given stochastic basis, construct a family of Libor rates $L(\cdot, T_k)$ and a collection of mutually equivalent probability measures $\mathbb{P}_{T_k}$ such that

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- in addition model defaultable Libor rates $L_C(\cdot, T_k)$ such that

$$\left(\frac{B_C(t, T_j)}{B(t, T_k)}\right)_{0 \le t \le T_k \wedge T_j}$$

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Defaultable bonds with ratings

- Credit ratings identified with elements of a finite set $\mathcal{K} = \{1, 2, \dots, K\}$, where 1 is the best possible rating and K is the default event
- Credit migration is modeled by a conditional Markov chain C with state space $\mathcal{K}$
- Default time τ: the first time when C reaches the absorbing state K, i.e.

$$\tau = \inf\{t > 0 : C_t = K\}$$

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- We consider defaultable bonds $B_C(\cdot, T_k)$ with credit migration process C and fractional recovery of Treasury value $q = (q_1, \ldots, q_{K-1})$ upon default:

$$B_C(t, T_k) = \sum_{i=1}^{K-1} B_i(t, T_k) \mathbf{1}_{\{C_t = i\}} + q_{C_{\tau-}} B(t, T_k) \mathbf{1}_{\{C_t = K\}},$$

We have $B_i(T_k, T_k) = 1$, for all i.

Canonical construction of C

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}, \mathbb{P}_{T^*})$ be a given complete stochastic basis.

- Let $\Lambda = (\Lambda_t)_{0 \le t \le T^*}$ be a matrix-valued $\mathbb{F}$-adapted stochastic process

$$\Lambda(t) = \begin{bmatrix} \lambda_{11}(t) & \lambda_{12}(t) & \dots & \lambda_{1K}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) & \dots & \lambda_{2K}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

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- Enlarge probability space

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The process C is a *conditional Markov chain* relative to $\mathbb{F}$, i.e. for every $0 \le t \le s$ and any function $h : \mathcal{K} \to \mathbb{R}$

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s) | \mathcal{F}_t \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s) | \mathcal{F}_t \vee \sigma(C_t)],$$

where $\mathbb{F}^C = (\mathcal{F}_t^C)$ denotes the filtration generated by C.

The progressive enlargement of filtration

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_t^C, \quad t \in [0, T^*],$$

satisfies the $(\mathcal{H})$-hypothesis:

$(\mathcal{H})$ Every local $\mathbb{F}$-martingale is a local $\mathbb{G}$-martingale.

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$(\mathcal{H}1)$ $\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_{T^*}] = \mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_t]$,

for any bounded, $\mathcal{F}_t^C$-measurable random variable Y (Brémaud and Yor (1978), Elliot, Jeanblanc and Yor (2000))

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But this follows easily from property

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_B|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_B|\mathcal{F}_t], \qquad t \leq s, B \in \mathcal{F}_t^C,$$

which is proved as a consequence of the canonical construction.

Risk-free Lévy Libor model

(Eberlein and Özkan, 2005)

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}, \mathbb{P}_{T^*})$ be a complete stochastic basis.

- as driving process take a time-inhomogeneous Lévy process $X = (X^1, \ldots, X^d)$ whose Lévy measures satisfy certain integrability conditions
- X is a special semimartingale with canonical decomposition

$$X_t = \int_0^t b_s \mathrm{d}s + \int_0^t \sqrt{c_s} \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x),$$

where W^{T^*} denotes a $\mathbb{P}_{T^*}$-standard Brownian motion and
μ is the random measure of jumps of X with $\mathbb{P}_{T^*}$-compensator ν^{T^*}.
We assume that $b = 0$.

Construction of Libor rates (backward induction):

Starting from $k = n-1$, we have for each T_k:

(i) define the forward measure $\mathbb{P}_{T_{k+1}}$ via

$$\left.\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T^*}}\right|_{\mathcal{F}_t} = \prod_{l=k+1}^{n-1} \frac{1+\delta_l L(t,T_l)}{1+\delta_l L(0,T_l)} = \frac{B(0,T^*)}{B(0,T_{k+1})} \frac{B(t,T_{k+1})}{B(t,T^*)}.$$

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(ii) the dynamics of the Libor rate $L(\cdot,T_k)$ under this measure

$$L(t,T_k) = L(0,T_k)\exp\left(\int_0^t b^L(s,T_k)\mathrm{d}s + \int_0^t \sigma(s,T_k)\mathrm{d}X_s^{T_{k+1}}\right), \tag{1}$$

where

$$X_t^{T_{k+1}} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T_{k+1}} + \int_0^t\int_{\mathbb{R}^d} x(\mu-\nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)$$

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with $\mathbb{P}_{T_{k+1}}$-Brownian motion $W^{T_{k+1}}$ and

$$\nu^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x) = \prod_{l=k+1}^{n-1}\left(\frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}(e^{\langle\sigma(s,T_l),x\rangle}-1)+1\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x).$$

The drift term $b^L(s,T_k)$ is chosen such that $L(\cdot,T_k)$ becomes a $\mathbb{P}_{T_{k+1}}$-martingale.

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(6) Define on this space the forward measures $\mathbb{Q}_{T_k}$ by:

for each tenor date T_k $\mathbb{Q}_{T_k}$ is obtained from $\mathbb{Q}_{T^*}$ in the same way as $\mathbb{P}_{T_k}$ from $\mathbb{P}_{T^*}$ $(k = 1, \ldots, n-1)$

Conditional Markov chain C *under forward measures*

Note that
$$\frac{\mathrm{d}\mathbb{Q}_{T_k}}{\mathrm{d}\mathbb{Q}_{T^*}} = \psi^k,$$
where ψ^k is an $\mathcal{F}_{T_k}$-measurable random variable with expectation 1.

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Theorem

Let C be a canonically constructed conditional Markov chain with respect to $\mathbb{Q}_{T^*}$. Then C is a conditional Markov chain with respect to every forward measure $\mathbb{Q}_{T_k}$ and

$$p_{ij}^{\mathbb{Q}_{T_k}}(t,s) = p_{ij}^{\mathbb{Q}_{T^*}}(t,s)$$

i.e. the matrices of transition probabilities under $\mathbb{Q}_{T^*}$ and $\mathbb{Q}_{T_k}$ are the same.

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Theorem

The $(\mathcal{H})$-hypothesis holds under all $\mathbb{Q}_{T_k}$, i.e. every $(\mathbb{F}, \mathbb{Q}_{T_k})$-local martingale is a $(\mathbb{G}, \mathbb{Q}_{T_k})$-local martingale.

Rating-dependent Libor rates

- The forward Libor rate for credit rating class i

$$L_i(t, T_k) := \frac{1}{\delta_k}\left(\frac{B_i(t, T_k)}{B_i(t, T_{k+1})} - 1\right), \quad i = 1, 2, \ldots, K-1$$

We put $L_0(t, T_k) := L(t, T_k)$ (default-free Libor rates).

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- The corresponding discrete-tenor forward inter-rating spreads

$$H_i(t, T_k) := \frac{L_i(t, T_k) - L_{i-1}(t, T_k)}{1 + \delta_k L_{i-1}(t, T_k)}$$

Observe that the Libor rate for the rating i can be expressed as

$$\begin{aligned} 1 + \delta_k L_i(t, T_k) &= (1 + \delta_k L_{i-1}(t, T_k))(1 + \delta_k H_i(t, T_k)) \\ &= \underbrace{(1 + \delta_k L(t, T_k))}_{\text{default-free Libor}} \prod_{j=1}^{i} \underbrace{(1 + \delta_k H_j(t, T_k))}_{\text{spread } j-1 \to j} \end{aligned}$$

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\end{aligned}
$$

Idea: model $H_j(\cdot,T_k)$ as exponential semimartingales and thus ensure automatically the *monotonicity* of Libor rates w.r.t. the credit rating:

$$L(t,T_k) \le L_1(t,T_k) \le \cdots \le L_{K-1}(t,T_k)$$

$\Longrightarrow$ worse credit rating, higher interest rate

Pre-default term structure of rating-dependent Libor rates

For each rating i and tenor date T_k we model $H_i(\cdot, T_k)$ as

$$H_i(t, T_k) = H_i(0, T_k) \exp\left(\int_0^t b^{H_i}(s, T_k)\mathrm{d}s + \int_0^t \gamma_i(s, T_k)\mathrm{d}X_s^{T_{k+1}}\right) \tag{2}$$

with initial condition

$$H_i(0, T_k) = \frac{1}{\delta_k}\left(\frac{B_i(0, T_k)B_{i-1}(0, T_{k+1})}{B_{i-1}(0, T_k)B_i(0, T_{k+1})} - 1\right).$$

$X^{T_{k+1}}$ is defined as earlier and $b^{H_i}(s, T_k)$ is the drift term (we assume $b^{H_i}(s, T_k) = 0$, for $s > T_k \Rightarrow H_i(t, T_k) = H_i(T_k, T_k)$, for $t \geq T_k$).

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$\Rightarrow$ the forward Libor rate $L_i(\cdot, T_k)$ is obtained from relation

$$1 + \delta_k L_i(t, T_k) = (1 + \delta_k L(t, T_k)) \prod_{j=1}^{i} (1 + \delta_k H_j(t, T_k)).$$

Theorem

Assume that $L(\cdot, T_k)$ and $H_i(\cdot, T_k)$ are given by (1) and (2). Then:

(a) *The rating-dependent forward Libor rates satisfy for every T_k and $t \leq T_k$*

$$L(t, T_k) \leq L_1(t, T_k) \leq \cdots \leq L_{K-1}(t, T_k),$$

i.e. Libor rates are monotone with respect to credit ratings.

(b) *The dynamics of the Libor rate $L_i(\cdot, T_k)$ under $\mathbb{P}_{T_{k+1}}$ is given by*

$$\begin{aligned} L_i(t, T_k) = L_i(0, T_k) \exp\Bigg(\int_0^t b^{L_i}(s, T_k)\mathrm{d}s + \int_0^t \sqrt{c_s}\sigma_i(s, T_k)\mathrm{d}W_s^{T_{k+1}} \\ + \int_0^t \int_{\mathbb{R}^d} S_i(s, x, T_k)(\mu - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x) \Bigg), \end{aligned}$$

where

$$\sigma_i(s, T_k) := \ell_i(s-, T_k)^{-1}\Big(\ell_{i-1}(s-, T_k)\sigma_{i-1}(s, T_k) + h_i(s-, T_k)\gamma_i(s, T_k)\Big)$$
$$= \ell_i(s-, T_k)^{-1}\Big[\ell(s-, T_k)\sigma(s, T_k) + \sum_{j=1}^{i} h_j(s-, T_k)\gamma_j(s, T_k)\Big]$$

represents the volatility of the Brownian part and

$$S_i(s, x, T_k) := \ln\Big(1 + \ell_i(s-, T_k)^{-1}(\beta_i(s, x, T_k) - 1)\Big)$$

controls the jump size. Here we set

$$h_i(s, T_k) := \frac{\delta_k H_i(s, T_k)}{1 + \delta_k H_i(s, T_k)},$$
$$\ell_i(s, T_k) := \frac{\delta_k L_i(s, T_k)}{1 + \delta_k L_i(s, T_k)},$$

and

$$\beta_i(s, x, T_k) := \beta_{i-1}(s, x, T_k)\Big(1 + h_i(s-, T_k)(e^{\langle \gamma_i(s, T_k), x\rangle} - 1)\Big)$$
$$= \Big(1 + \ell(s-, T_k)(e^{\langle \sigma(s, T_k), x\rangle} - 1)\Big)$$
$$\times \prod_{j=1}^{i}\Big(1 + h_j(s-, T_k)(e^{\langle \gamma_j(s, T_k), x\rangle} - 1)\Big).$$

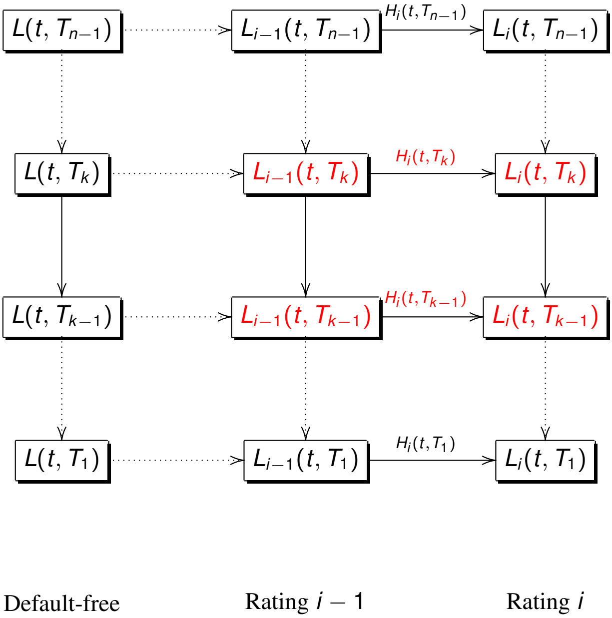

Figure: Connection between subsequent Libor rates

No-arbitrage condition for the rating based model

Recall the defaultable bond price process with fractional recovery of Treasury value q

$$B_C(t, T_k) = \sum_{i=1}^{K-1} B_i(t, T_k) \mathbf{1}_{\{C_t = i\}} + q_{C_{\tau\_}} B(t, T_k) \mathbf{1}_{\{C_t = K\}}.$$

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Note: the forward bond price process

$$\frac{B_C(\cdot, T_k)}{B(\cdot, T_j)}$$

is a $\mathbb{Q}_{T_j}$-local martingale for every $k, j = 1, \ldots, n-1$

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is a $\mathbb{Q}_{T_k}$-local martingale for every $k = 1,\ldots,n-1$.

We postulate that the forward bond price process is given by

$$\frac{B_C(t,T_k)}{B(t,T_k)} := \sum_{i=1}^{K-1} \underbrace{\prod_{j=1}^{i}\prod_{l=0}^{k-1} \frac{1}{1+\delta_l H_j(t,T_l)}}_{:=\mathbb{H}(t,T_k,i)} e^{\int_0^t \lambda_i(s)\mathrm{d}s} \mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}} \mathbf{1}_{\{C_t=K\}}$$

$$= \sum_{i=1}^{K-1} \mathbb{H}(t,T_k,i) e^{\int_0^t \lambda_i(s)\mathrm{d}s} \mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}} \mathbf{1}_{\{C_t=K\}}, \tag{3}$$

where λ_i is some $\mathbb{F}$-adapted process that is integrable on $[0, T^*]$. (go to DFM)

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where λ_i is some $\mathbb{F}$-adapted process that is integrable on $[0,T^*]$. (go to DFM)

Note that this specification is consistent with the definition of H_i which implies the following connection of bond prices and inter-rating spreads:

$$\frac{B_j(t,T_k)}{B_{j-1}(t,T_k)} = \frac{B_j(t,T_{k-1})}{B_{j-1}(t,T_{k-1})} \frac{1}{1+\delta_{k-1}H_j(t,T_{k-1})}$$

and relation

$$\frac{B_i(t,T_k)}{B(t,T_k)} = \frac{B_1(t,T_k)}{B(t,T_k)} \prod_{j=2}^{i} \frac{B_j(t,T_k)}{B_{j-1}(t,T_k)}.$$

Lemma

Let T_k be a tenor date and assume that $H_j(\cdot, T_k)$ are given by (2)*. The process $\mathbb{H}(\cdot, T_k, i)$ has the following dynamics under $\mathbb{P}_{T_k}$*

$$
\begin{aligned}
\mathbb{H}(t, T_k, i) &= \mathbb{H}(0, T_k, i) \\
&\quad \times \mathcal{E}_t \Bigg(\int_0^{\cdot} b^{\mathbb{H}}(s, T_k, i)\mathrm{d}s - \int_0^{\cdot} \sqrt{c_s} \sum_{j=1}^{i} \sum_{l=1}^{k-1} h_j(s-, T_l)\gamma_j(s, T_l)\mathrm{d}W_s^{T_k} \\
&\qquad + \int_0^{\cdot} \int_{\mathbb{R}^d} \left(\prod_{j=1}^{i} \prod_{l=1}^{k-1} \left(1 + h_j(s-, T_l)(e^{\langle \gamma_j(s, T_l), x\rangle} - 1)\right)^{-1} - 1 \right) \\
&\qquad\qquad \times (\mu - \nu^{T_k})(\mathrm{d}s, \mathrm{d}x) \Bigg),
\end{aligned}
$$

where $b^{\mathbb{H}}(s, T_k, i)$ is the drift term.

No-arbitrage condition

Theorem

Let T_k be a tenor date. Assume that the processes $H_j(\cdot, T_k)$, $j = 1, \ldots, K-1$, are given by (2). Then the process $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ defined in (3) is a local martingale with respect to the forward measure $\mathbb{Q}_{T_k}$ and filtration $\mathbb{G}$ iff: for almost all $t \leq T_k$ on the set $\{C_t \neq K\}$

$$b^{\mathbb{H}}(t, T_k, C_t) + \lambda_{C_t}(t) = \left(1 - q_{C_t}\frac{e^{-\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}{\mathbb{H}(t-, T_k, C_t)}\right)\lambda_{C_t K}(t) + \sum_{j=1, j\neq C_t}^{K-1}\left(1 - \frac{\mathbb{H}(t-, T_k, j)e^{\int_0^t \lambda_j(s)\mathrm{d}s}}{\mathbb{H}(t-, T_k, C_t)e^{\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}\right)\lambda_{C_t j}(t). \tag{4}$$

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Sketch of the proof: Use the fact that the jump times of the conditional Markov chain C do not coincide with the jumps of any $\mathbb{F}$-adapted semimartingale, use martingales related to the indicator processes $\mathbf{1}_{\{C_t = i\}}$, $i \in \mathcal{K}$, and stochastic calculus for semimartingales.

Defaultable forward measures

Assume that $\frac{B_C(\cdot, T_k)}{B(\cdot, T_k)}$ is a *true martingale* w.r.t. forward measure $\mathbb{Q}_{T_k}$. (back to DFP)

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The defaultable forward measure $\mathbb{Q}_{C,T_k}$ for the date T_k is defined on $(\Omega, \mathcal{G}_{T_k})$ by

$$\left.\frac{\mathrm{d}\mathbb{Q}_{C,T_k}}{\mathrm{d}\mathbb{Q}_{T_k}}\right|_{\mathcal{G}_t} := \frac{B(0,T_k)}{B_C(0,T_k)}\,\frac{B_C(t,T_k)}{B(t,T_k)}.$$

This corresponds to the choice of $B_C(\cdot, T_k)$ as a numeraire.

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Proposition

The defaultable Libor rate $L_C(\cdot,T_k)$ is a martingale with respect to $\mathbb{Q}_{C,T_{k+1}}$ and

$$\left.\frac{\mathrm{d}\mathbb{Q}_{C,T_k}}{\mathrm{d}\mathbb{Q}_{C,T_{k+1}}}\right|_{\mathcal{G}_t} = \frac{B_C(0,T_{k+1})}{B_C(0,T_k)}(1+\delta_k L_C(t,T_k)).$$

Pricing problems I: Defaultable bond

Proposition

The price of a defaultable bond with maturity T_k and fractional recovery of Treasury value q at time $t \leq T_k$ is given by

$$
\begin{aligned}
B_C(t, T_k)\mathbf{1}_{\{C_t \neq K\}} = B(t, T_k) \sum_{i=1}^{K-1} \mathbf{1}_{\{C_t = i\}} \Bigg[& \mathbb{E}_{\mathbb{Q}_{T_k}}[1 - p_{iK}(t, T_k) | \mathcal{F}_t] \\
& + \sum_{j=1}^{K-1} \frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{t < \tau \leq T_k\}} \mathbf{1}_{\{C_t = i\}} \mathbf{1}_{\{C_{\tau-} = j\}} q_j | \mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t = i\}} | \mathcal{F}_t]} \Bigg].
\end{aligned}
$$

Pricing problems II: Credit default swap

- consider a maturity date T_m and a defaultable bond with fractional recovery of Treasury value q as the underlying asset
- protection buyer pays a fixed amount S periodically at tenor dates $T_1, \ldots, T_{m-1}$ until default
- protection seller promises to make a payment that covers the loss if default happens:

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has to paid at T_{k+1} if default occurs in $(T_k, T_{k+1}]$

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Proposition

The swap rate S at time 0 is equal to

$$S = \frac{\sum_{k=2}^{m} B(0, T_k) \sum_{j=1}^{K-1} \mathbb{E}_{\mathbb{Q}_{T_k}}[(1 - q_j)\mathbf{1}_{\{T_{k-1} < \tau \leq T_k, C_{\tau-} = j\}}]}{\sum_{k=1}^{m-1} B(0, T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[1 - p_{iK}(0, T_k)]},$$

if the observed class at time zero is i.

Pricing problems III: use of defaultable measures

Proposition

Let Y be a promised $\mathcal{G}_{T_k}$-measurable payoff at maturity T_k of a defaultable contingent claim with fractional recovery q upon default and assume that Y is integrable with respect to $\mathbb{Q}_{T_k}$.
The time-t value of such a claim is given by

$$\pi^t(Y) = B_C(t, T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[Y|\mathcal{G}_t].$$

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Example: a cap on the defaultable forward Libor rate

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Example: a cap on the defaultable forward Libor rate

The time-t price of a caplet with strike K and maturity T_k on the defaultable Libor rate is given by

$$C_t(T_k, K) = \delta_k B_C(t, T_{k+1})\mathbb{E}_{\mathbb{Q}_{C,T_{k+1}}}[(L_C(T_k, T_k) - K)^+|\mathcal{G}_t]$$

and the price of the defaultable forward Libor rate cap at time $t \leq T_1$ is given as a sum

$$\mathbb{C}_t(K) = \sum_{k=1}^{n} \delta_{k-1} B_C(t, T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[(L_C(T_{k-1}, T_{k-1}) - K)^+|\mathcal{G}_t].$$

Concluding remarks

- this model provides a way to include credit risk with ratings in the Libor market models
- as driving processes a wide class of time-inhomogeneous Lévy processes is used
- extensions are possible to portfolio credit risk modeling (Eberlein, Grbac, Schmidt (2010))
- similar approach could be used for modeling of variations in the credit quality of the Libor contributing banks

The talk is based on:

E. Eberlein and Z. Grbac, Rating-based Lévy Libor model, Preprint, University of Freiburg, 2010. (submitted)