Drifting sub-Riemannian fronts: classifications and singularities

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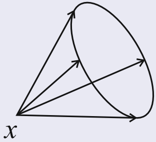
Drifted sub-Riemannian control system in X^3:

- Admissible velocities form a distribution of ellipses:

$$\mathrm{Ell}_{\mathbf{x}} = \{a(\mathbf{x})\cos\varphi + b(\mathbf{x})\sin\varphi + c(\mathbf{x})\} \subset T_{\mathbf{x}}X^3, \quad a, b, c \in \mathrm{Vect}(X^3);$$

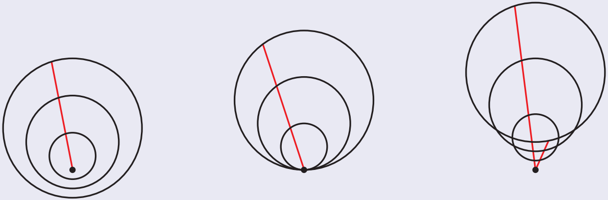


- admissible trajectories: $\dot\gamma(t) \in \mathrm{Ell}_{\gamma(t)}$
- locally extremal admissible trajectories (extremal trajectories): $\gamma(0) = A, \quad \gamma(T) = B, \quad \delta T = 0;$
- drifting sub-Riemannian front:

$$S^r_A = \{\gamma(r) \in X^3 \mid \gamma(0) = A,\ \gamma \text{ is extremal}\}.$$

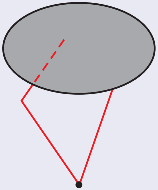
Examples:

- Plane case (drifting circle): $a = (1,0,0),\ b = (0,1,0),\ c = (0,\lambda,0)$



Three possibilities $0 < \lambda < 1,\ \lambda = 1,\ \lambda > 1$.

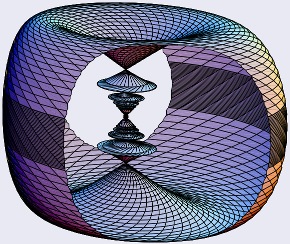
- Drifting disk: $a = (1,0,0),\ b = (0,1,0),\ c = (0,0,1)$



All admissible trajectories are extremal!!!

Sub-Riemannian front

- Degenerate case: $a = (1, 0, -y)$, $b = (0, 1, x)$, $c = 0$
  Sub-Riemannian structure: $x\,dy - y\,dx - dz = 0$, $ds^2 = dx^2 + dy^2$
  Degenerate $=$
  (1) quasihomogeneous with $\deg x = \deg y = 1$, $\deg z = 2$
  $+$
  (2) (x, y)-rotation invariant.
  The equations are integrable explicitly.



- Non-degenerate (non-quasihomogeneous, non-integrable) case is more complicated and I do not have a figure.

Degenerate contact hyperbolic drifting sub-Riemannian front

$a = (1, 0, 0),\ b = (0, 1, 0),\ c = (0, 0, x)$

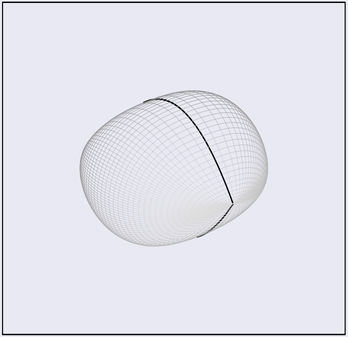
Degenerate =

(1) quasihomogeneous with $\deg x = \deg y = 1,\ \deg z = 2$

+

(2) hyperbolic-rotation invariant what will be explained later.

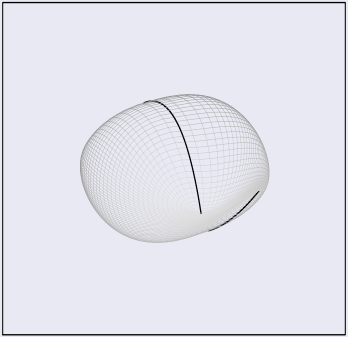
The equations are integrable explicitly.



Theorem 1

If a, b, c are generic then for almost every point $A \in X^3$ there are only two possibilities:

1. A is contact hyperbolic and S_A^ε is the non-degenerate contact hyperbolic drifting sub-Riemannian front for sufficiently small ε:



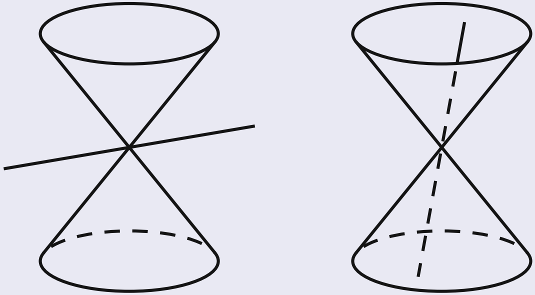
2. A is contact elliptic.

Conjecture

If a, b, c are generic and for almost every contact elliptic point S_A^ε is the non-degenerate sub-Riemannian front for sufficiently small ε.

Contact hyperbolic and contact elliptic points

- $M^4 = X^3 \times \mathbb{R}$;
- $H^3_m = \langle (a(\mathbf{x}), 1), (b(\mathbf{x}), 1), (c(\mathbf{x}), 1) \rangle \subset T_m M^4,\ m = (\mathbf{x}, t)$;
- $H^3_m = \ker \theta(m)$, θ is 1-form on M,
  let $K^1_m \subset H^3_m,\ d\theta(m)(K^1_m, H^3_m) = 0$;
- $\mathrm{Ell}_{\mathbf{x}}$ defines $\mathrm{Cone}^2_m \subset H^3_m \subset T_m M^4$;
- Cone^2_m and K^1_m in H^3_m:



Contact hyperbolic and contact elliptic points

Example:

For a sub-Riemannian structure K^1 is generated by the vector $\dot{\mathbf{x}} = 0$, $\dot{t} = 1$, Cone^2 is $dt^2 = ds^2$, and all points are contact elliptic.

Theorem 2

In a neighborhood of a contact hyperbolic point in M^4 there exist local coordinates (u, v, w, z) such that the our control system is described by the equations:

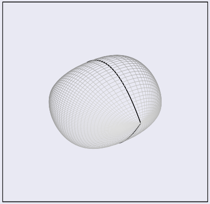
$$du^2 - dv\,dw + \alpha\,u\,dv^2 + \beta\,u\,dw^2 + \ldots = 0, \quad \alpha, \beta \in \mathbb{R},$$

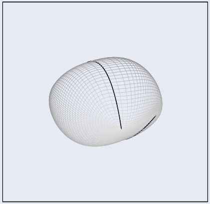
$$dz = v\,dw - w\,dv,$$

where $\alpha = 0$ or 1 and $\ldots$ are terms of higher quasihomogeneous degree if $\deg u = \deg v = \deg w = 1$, $\deg z = 2$.

Cases:

- Degenerate (twice-quasihomogeneous) $du^2 - dv\,dw = 0$, $dz = v\,dw - w\,dv$, $\deg(u,v,w,z) = (1,1,1,2)$, $= (0,1,-1,0)$; this case is integrable and analogous to the degenerate sub-Riemannian case in the contact elliptic situation.
- Half-degenerate (quasihomogeneous) $du^2 - dv\,dw + u\,dv^2 = 0$, $dz = v\,dw - w\,dv$, $\deg(u,v,w,z) = (2,1,3,4)$; this case is integrable too and does not have analogy in the contact elliptic situation.
- Non-degenerate (non-quasihomogeneous) $\alpha = 1$, $\beta \neq 0$, this case is non-integrable and described by Theorem 1.





Connection with two Arnold's normal forms

- $\mathrm{Ell}_m = P\mathrm{Cone}_m^2 \subset PT_mM^4$ is a plane quadric (conic) lying in the projective plane PH_m^3;
- $\mathrm{Ell}_m^* \subset PT_m^*M^4$ is (again!) a two-dimensional cone with a vertex $PH_m^3 \in PT_m^*M^4$;
- $\Sigma^6 \subset PT^*M^4$, $\Sigma \cap PT_m^*M^4 = \mathrm{Ell}_m^*$;
- $M^4 = X^3 \times \mathbb{R}$, $PT^*M^4 \supset J^1(X, \mathbb{R}) \cong T^*X \times \mathbb{R}$,
  $a, b, c : T^*X \to \mathbb{R}$, $\Sigma = \{a^2 + b^2 - (c-1)^2 = 0\}$
  $a = (1,0,0),\ b = (0,1,0),\ c = (0,0,x),\ p^2 + q^2 - (rx-1)^2 = 1$
  $a = (1,0,-y),\ b = (0,1,x),\ c = 0,\ (p - ry)^2 + (q + rx)^2 = 1$
- two Arnold's local normal forms for Σ in a neighborhood of PH_m w. r. t. contact diffeomorphisms of $PT_m^*M^4$:

$$p_1^2 \pm q_1^2 - p_2^2 = 0,$$

$$p_1\, dq_1 - q_1\, dp_1 + \cdots + p_3\, dq_3 - q_3\, dp_3 - 2du = 0;$$

- $+ =$ contact ellipticity, $- =$ contact hyperbolicity.

Connection with systems of linear PDE

$$\begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix} = \begin{pmatrix} \partial_a & 0 \\ 0 & -\partial_a \end{pmatrix} + \begin{pmatrix} 0 & \partial_b \\ \partial_b & 0 \end{pmatrix} + \begin{pmatrix} \partial_c & 0 \\ 0 & \partial_c \end{pmatrix}$$

$$J^1(X,\mathbb{R}) \cong T^*X \times \mathbb{R}, \quad a,b,c : T^*X \to \mathbb{R},$$

$$\sigma = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Sigma = \{\det \sigma\} = \{a^2 + b^2 - (c-1)^2 = 0\}$$

Examples:

- $a = (1,0,0),\ b = (0,1,0),\ c = (0,0,x),$
  $\partial_a = \partial_x,\ \partial_b = \partial_y,\ \partial_c = x\partial_z$
- $a = (1,0,-y),\ b = (0,1,x),\ c = 0,$
  $\partial_a = \partial_x - y\partial_z,\ \partial_b = \partial_y + x\partial_z,\ \partial_c = 0$

CONGRATULATIONS TO ANDREY !!!