DETERMINISTIC MEAN FIELD GAMES

Italo Capuzzo Dolcetta
Sapienza Università di Roma
and
GNAMPA - Istituto di Alta Matematica

A classical optimization problem

Given a time interval $[0, T]$ consider the classical Mayer type problem

$$\inf \int_t^T \left[\frac{1}{2}|\dot{X}_s|^2 + L(X_s)\right] ds + G(X_T) \tag{1}$$

where $X := X^{t,x}$ is any curve in the Sobolev space $W^{1,2}([t, T]; \mathbb{R}^d)$ such that $X_T = x \in \mathbb{R}^d$ for $t \in [0, T]$.

Well-known that if $L : \mathbb{R}^d \times [0, T] \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ are continuous and bounded, then the **value function** of problem (1) above, i.e.

$$u(t, x) = \inf \left\{ \int_t^T \left[\frac{1}{2}|\dot{X}_s|^2 + L(X_s)\right] ds + G(X_T)\,;\; X \in W^{1,2}([0, T]; \mathbb{R}^d) \right\}$$

is the unique bounded continuous **viscosity solution** of the backward Cauchy problem HJ

$$\begin{cases} -\partial_t u(t,x) + \frac{1}{2}\left|\nabla_x u(t,x)\right|^2 = L(x) & \text{in } (0,T)\times\mathbb{R}^d, \\ u(T,x) = G(x) & \text{in } \mathbb{R}^d \end{cases} \tag{2}$$

of Hamilton-Jacobi type.
The proof that u solves (2) in viscosity sense is a simple consequence of the following identity, the **Dynamic Programming Principle**:

$$u(t,x) = \inf\left\{ u(s, X^{t,x}(s)) + \int_s^t L(X_s)\,ds \; ; \quad X \in W^{1,2}([0,T];\mathbb{R}^d)\right\}$$

valid for any given $(t,x) \in (0,T)\times\mathbb{R}^d$ and any $s \in [t,T]$.

Uniqueness of solution is a non trivial, fundamental result in viscosity solutions theory (Lions 1982).

As for **optimal curves**, easy to check that $\overline{X}^{t,x}$ is optimal for the initial setting (t,x) if and only if

$$u(t,x) = u(s,\overline{X}^{t,x}(s)) + \int_s^T L(\overline{X}^{t,x}(\tau))\,d\tau \quad \text{for all} \quad s \in [t,T]$$

Moreover, if u is smooth enough, the **velocity field** of the **optimal paths** is the spatial gradient of the solution of the HJ equation.

More precisely,

A Verification Lemma

Lemma

Let $X^*(t)$ *be such that*

$$\dot{X}^*(s) = -\nabla_x u(s, X^*(s)) \ \ \textit{for} \ \ s \in [t, T] \ \ , \ \ X^*(t) = x$$

Then,

$$\int_t^T \left[\frac{1}{2}|\dot{X}^*(s)|^2 + L(X^*(s))\right] ds + G(X^*(T)) =$$

$$= \inf \int_t^T \left[\frac{1}{2}|\dot{X}_s|^2 + L(X_s)\right] ds + G(X_T)$$

Verification result above requires u to be C^1 with respect to x. This turns out to be true in the present model problem under a C^2 smoothness assumptions on L, G.
The proof of C^1 regularity of u is in 3 steps:

- step 1: u is globally Lipschitz w.r.t (t, x)
- step 2 : u is semiconcave w.r.t. x, i.e. $x \to u(t,x) - \frac{1}{2}C_t|x|^2$ concave for some positive constant C_t
- step 3: the upper semidifferential

$$D_x^+ u(t,x) = \left\{ p \in \mathbb{R}^d : \limsup_{y \to x} \frac{u(t,y) - u(t,x) - p \cdot (y-x)}{|y-x|} \leq 0 \right\}$$

is a singleton at each (t,x)

Alternative way to optimal feebacks for general control problems when no smoothness available is via semi-discretization (comments on this issue later on)

Proof of Verification Lemma:

$$u(T, X_T) = u(t, X_T) + \int_t^T \left[\partial_s u(s, X_s) + \dot{X}_s \cdot \nabla u(s, X_s)\right] ds =$$

[by HJ]

$$= u(t, X_T) + \int_t^T \left[\frac{1}{2}|\nabla_x u(s, X_s)|^2 + \dot{X}_s \cdot \nabla_x u(s, X_s) - L(X_s)\right] ds \geq$$

[by **convexity of** $p \to \frac{1}{2}|p|^2$]

$$\geq u(t, X_T) + \int_t^T \left[-\frac{1}{2}|\dot{X}_s|^2 - L(X_s)\right] ds$$

Since $u(T, X_T) = G(X_T)$, $u(t, X_T) = u(t, x)$, above yields

$$G(X_T) + \int_t^T \left[\frac{1}{2}|\dot{X}_s|^2 + L(X_s)\right] ds \geq u(t, x)$$

Same computation with generic curve X replaced by X^* given by

$$\dot{X}^*(s) = -\nabla_x u(s, X^*(s)) \;\text{ for }\; s \in [t, T] \;\;,\;\; X^*(t) = x$$

gives $=$ in the last step, so that

$$u(t, x) = \inf \int_t^T \left[\frac{1}{2}|\dot{X}_s|^2 + L(X_s)\right] ds + G(X_T)$$

A deterministic mean field game problem

An interesting new class of optimal control has become recently object of interest after the 2006/07 papers by Lasry and Lions (see also P.-L. Lions, Cours au Collège de France www.college-de-france.fr. for more recent developments)

Related ideas have been developed independently in the engineering literature, and at about the same time, by Huang, Caines and Malhamé.

Assume that the running cost $L(X_s)$ depends also on an **exhogenous variable** $m(s, X_s)$ modeling the **density of population** of the other agents at state X_s at time s.

The new cost criterion is then

$$\inf \int_t^T \left[\frac{1}{2}|\dot{X}_s|^2 + L(X_s, m(s, X_s))\right] ds + G(X_T, m(T, X_T)) \qquad (3)$$

Here, m is a **non-negative** function valued in $[0,1]$ such that $\int_{\mathbb{R}^d} m(s,x)\,dx = 1$ for all s.

The time evolution of m starting from an initial configuration $m(0,x)$ is governed by the **continuity equation**

$$\partial_t m(t,x) - div\,(m(t,x) D_x u(t,x)) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d$$

Note that in the cost criterion the evolution of the measure m enters as a parameter. The value function of the agent is then given by

$$\inf \int_t^T \left[\frac{1}{2}|\dot{X}_s|^2 + L(X_s, m(s, X_s))\right] ds + G(X_T, m(T, X_T)) \qquad (4)$$

His optimal control is, at least heuristically, given in feedback form by $\alpha^*(t,x) = -\nabla_x u(t,x)$.

Now, if all agents argue in this way, their repartition will move with a velocity which is due to the drift term $\nabla_x u(t,x)$.
This leads eventually to the continuity equation.

We are therefore led to consider the following system of nonlinear evolution pde's for the unknown functions $u = u(t,x)$, $m = m(t,x)$:

$$-\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = L(x,m) \quad \text{in } (0,T) \times \mathbb{R}^d \tag{5}$$

$$\frac{\partial m}{\partial t} - div\,(m\,\nabla u) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d \tag{6}$$

with the initial and terminal conditions

$$m(0,x) = m_0(x), \quad u(T,x) = G(x, m(T,x)) \quad \text{in } \mathbb{R}^d \tag{7}$$

Three crucial structural features:

- first equation **backward**, second one **forward** in time
- the operator in the **continuity** equation is the **adjoint of the linearization** at u of the operator in the HJ operator in the first equation
- nonlinearity in the HJB equation is **convex** with respect to $|\nabla u|$

The planning problem

An interesting variant of the **MFG** system proposed by Lions for modeling the presence of a **regulator prescribing a target density to be reached at final time**:

$$\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = L(x, m) \quad \text{in } (0, T) \times \mathbb{R}^d$$

$$\frac{\partial m}{\partial t} - div\,(m\,\nabla u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d$$

with the initial and terminal conditions

$$m(0, x) = m_0(x) \geq 0, \quad m(T, x) = m_T(x), \quad \text{in } \mathbb{R}^d$$

No side conditions on u.

For $L \equiv 0$, the above is the equivalent formulation of **Monge-Kantorovich optimal mass transport** problem considered by Benamou-Brenier (2000), see also Achdou-Camilli-CD SIAM J. Control Optim. (2011).

Stochastic mean field game models

Consider the following system **(MFG)** of evolution pde's:

$$-\frac{\partial u}{\partial t} - \nu \Delta u + \frac{1}{2}|\nabla u|^2 = L(x, m) \quad \text{in } (0, T) \times \mathbb{R}^d \tag{8}$$

$$\frac{\partial m}{\partial t} - \nu \Delta m - div(m \nabla u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d \tag{9}$$

with the initial and terminal conditions

$$m(0, x) = m_0(x), \quad u(T, x) = G(x, m(T, x)) \quad \text{in } \mathbb{R}^d \tag{10}$$

ν is a positive number.

First equation is a backward **HJB** , the second one a forward **FP**

The heuristic interpretation of this system is as follows.
Fix a solution of **MFG** : classical **dynamic programming** approach to optimal control suggest that the solution u of **(HJB)** is the **value function** of an agent controlling the stochastic ODE

$$dX_t = \alpha_t \, dt + \sqrt{2\,\nu} \, dB_t, \quad X_0 = x$$

where B_t is a standard Brownian motion, i.e.

$$X_t = x + \int_0^t \alpha_s \, ds + \sqrt{2\,\nu} \, B_t$$

The agent aims at minimizing the integral cost

$$J(x,\alpha) := \mathbb{E}_x\Big[\int_0^T \left(\frac{1}{2}|\alpha_s|^2 + L(X_s, m(s)\right) \, ds + G(X_T, m(T))\Big]$$

considering the density $m(s)$ of "the other agents" as given.

Formal dynamic programming arguments indicate that the candidate optimal control for the agent should be constructed through the **feedback strategy** $\alpha^*(t,x) := -\nabla u(t,x)$ where u is the unique solution of **HJB** for fixed m.
Indeed, we have the simple **verification** result:

Lemma

Let X_t^* be the solution of

$$dX_t = \alpha^*(t, X_t)\, dt + \sqrt{2\,\nu}\, dB_t, \quad X_0 = x$$

and set $\alpha_t^* := \alpha^*(t, X_t)$. Then,

$$\inf_{\alpha} J(x,\alpha) = J(x,\alpha_t^*) = \int_{\mathbb{R}^d} u(0, X_0)\, dm_0(x)$$

Therefore, optimal control problem "completely" solved by solving backward **HJB** , determining $\nabla u(t,x)$ for all t and initial value $u(0,x)$

Proof:
Take $\nu = 1$ for simplicity and let α_t be any admissible control. Then,

$$\mathbb{E}_x\Big[G(X_T, m(T))\Big] = \mathbb{E}\Big[u(X_T, m(T))\Big] =$$

[by Ito's formula]

$$= \mathbb{E}_x\Big[u(0, X_0) + \int_0^T \Big(\frac{\partial u(s, X_s)}{\partial t} + \alpha_s \cdot \nabla u(s, X_s) + \Delta u(s, X_s) ds\Big)\Big] =$$

[by **HJB**]

$$= \mathbb{E}_x\Big[u(0, X_0) + \int_0^T \Big(\frac{1}{2}|\nabla u(s, X_s)|^2 + \alpha_s \cdot \nabla u(s, X_s) - F(X_s, m(s))\Big)\Big] \geq$$

[by convexity]

$$\geq \mathbb{E}_x\Big[u(0, X_0) + \int_0^T (-\frac{1}{2}|\alpha_s|^2 - L(X_s, m(s)))ds\Big]$$

Hence, by very definition of J,

$$\mathbb{E}_x\Big[u(0,x)\Big] \leq J(\alpha, x)$$

for any admissible control α.
The same computation with α_s replaced by α_s^* gives an equality in the last step, proving that

$$\inf_\alpha J(x,\alpha) = J(x,\alpha^*)$$

Above system is a simplified version of more general system introduced by Lasry-Lions (2006):

$$-\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla_x u) = L(x, m) \quad \text{in } (0, T) \times \mathbb{R}^d \tag{11}$$

$$\frac{\partial m}{\partial t} - \nu \Delta m - div\ (m \nabla_p H(x, \nabla_x u)) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d \tag{12}$$

with general convex function $p \to H(x, m, p)$.
In this more general case the cost functional is

$$J(x, \alpha) := \mathbb{E}_x \Big[\int_0^T (H^*(X_s, \alpha_s) + L(X_s, m(s))\ ds + G(X_T, m(T)) \Big]$$

where H^* is **Legendre-Fenchel transform** of the Hamiltonian H.
The crucial inequality

$$\mathbb{E}_x \Big[u(0, x) \Big] \le J(\alpha, x)$$

in the Verification Lemma is indeed an immediate consequence of the definition of the **LF** transform.

A few comments on models and directions of investigation:

- nonlocal operators: $L(x, m) = \int_{\mathbb{R}^d} K(x, y)m(y)\, dy$, Lasry-Lions (2007)
- degenerate diffusions: $\nu\Delta$ replaced by $\mathrm{Tr}(A(x)D^2)$ with $A(x)$ positive semidefinite, CD-Leoni-Porretta in progress,
- analysis of finite difference schemes, Achdou-CD SINUM (2010), Achdou-Camilli-CD SICON(2011), Achdou-Camilli-CD preprint (2012)
- switching problems Achdou-Camilli-CD , in progress
- optimal stopping time, obstacle problem in **HJB** ?
- fractional Laplacians instead of $\nu\Delta$?

Nash equilibria for differential games with N players and the MFG system

Let $J^i = J^i(\alpha^1, ..., \alpha^N)$ be real valued functionals defined on a product space $\mathcal{A}^{\mathcal{N}} = \mathcal{A} \times ... \times \mathcal{A}$. An N-tuple $(\overline{\alpha}^1, ..., \overline{\alpha}^N) \in \mathcal{A}^{\mathcal{N}}$ is a **Nash equilibrium** (Nash PNAS 1950) for the J^i's if

$$J^i(\overline{\alpha}^1, ..., \overline{\alpha}^N) \leq J^i(\overline{\alpha}^1, ..., \overline{\alpha}^{(i-1)}, \alpha^i, \overline{\alpha}^{(i+1)}, ..., \overline{\alpha}^N)$$

for each $i = 1, ..., N$ and each $\alpha^i \in \mathcal{A}$.

- existence of Nash equilibria in the space of measures (**randomized strategies**) (Nash PNAS 1950) can be proved by Ky Fan fixed point theorem
- no uniqueness in general
- dynamic programming optimality conditions: highly complex system of $2N$ nonlinear pde's in $2N$ unknown functions u_i (the value functions of the various players), see Bensoussan-Frehse (1980).

Consider N players whose state X_t^i, $i \in \{1, ..., N\}$, is given by

$$dX_t^i = \alpha_t^i dt + \sqrt{2\nu} dB_t^i, \quad 1 \leq i \leq N, , \quad t \in (0, +\infty)$$

$$X_0^i = x^i \in \mathbb{R}^d \, ,$$

α^i is the control of the i-th player, B_t^i independent Brownian motions
Assume that initial condition x^i are **random with a given probability law** m_0.
Each player has an individual cost functional of the **special form**:

$$J_N^i(x_i, \alpha_1, ..., \alpha_N) = \mathbb{E}_{x_i}\Big[\int_0^T \frac{1}{2}|\alpha_s^i|^2 + L(X_s^i, \frac{1}{N-1}\sum_{j \neq i} \delta_{X_s^j}) \, ds$$

$$+ G(X_T^i, \frac{1}{N-1}\sum_{j \neq i} \delta_{X_T^j}) \Big]$$

An interesting fact is that the verification procedure starting from the **MFG** system previously described provides an ϵ-Nash equilibrium for the above **"symmetric"** game which can be therefore interpreted as a sort of **discretized MFG**

The algorithm is as follows:

- take (u, m) (unique) solution of **MFG** :

$$-\frac{\partial u}{\partial t} - \nu \Delta u + \frac{1}{2}|\nabla u|^2 = L(x, m) \quad \text{in } (0, T) \times \mathbb{R}^d \tag{13}$$

$$\frac{\partial m}{\partial t} - \nu \Delta m - div\,(m \nabla u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d \tag{14}$$

  with the initial and terminal conditions

$$m(0, x) = m_0(x), \quad u(T, x) = G(x, m(T, x)) \quad \text{in } \mathbb{R}^d \tag{15}$$

- compute $\overline{\alpha}(t,x) := -\nabla u(t,x)$
- determine $\overline{X}_t^{*i}$ as the solution of the Ito's equation

$$d\overline{X}_t^i = \overline{\alpha}(t,\overline{X}_t^i) + \sqrt{2\,\nu}\,dB_t^i, \quad X_0^i = x^i$$

  where x^i are randomly distributed with the law m_0 (the initial condition in the **FP** equation)
- set $\overline{\alpha}^i = \overline{\alpha}^*(t, X^{*i})$

Next result says that the above synthesis procedure for **MFG** produces **almost optimal Nash equilibria** for the above described class of N players differential games provided N is **sufficiently large**:

Theorem

For any $\epsilon > 0$ *there is* N_ϵ *such that for* $N > N_\epsilon$

$$J^i(\overline{\alpha}^1, ..., \overline{\alpha}^N) \leq J^i(\overline{\alpha}^1, ..., \overline{\alpha}^{(i-1)}, \alpha^i, \overline{\alpha}^{(i+1)}, ..., \overline{\alpha}^N) + \epsilon$$

for each $i = 1, ..., N$ *and each* $\alpha^i \in \mathcal{A}$.

Technical proof based among other on Hewitt-Savage theorem (see Cardaliaguet).

Nash equilibria for N players as $N \to \infty$

Lions conjectured the above result to be **exact** (i.e. with $\epsilon = 0$) **in the limit** as $N \to +\infty$.

Classical **DP** approach (Bensoussan-Frehse 1984) to differential games with N players leads in fact to the consideration of a system of $2N$ quasilinear PDE's, a **highly complex** problem.

The validitation of such a conjecture would provide a rigorous implementation of dimension reduction to **simplified averaged models** comprising a system of just two pde's in the form of **MFG** .

This asymptotic result has been actually proved to be true, see Lasry-Lions (2007), under the same symmetry assumptions as above, in the case of **infinite horizon games for ergodic systems with compact state space**, namely the d- dimensional torus .

Bardi (to appear) has similar results, with detailed explicit computations, in the case of **linear-quadratic stochastic games**, see also Cardaliaguet (2010).

Weintraub, Benkard, Van Roy, Oblivious Equilibrium: A Mean Field Approximation for Large-Scale Dynamic Games, (discrete time Markov processes)

Other and/or more general cases: widely open.

A semi-discrete approach to deterministic MFG

We describe next a **semi-discretization** approach to the deterministic mean field game system:

$$-\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = L(x, m) \quad \text{in } (0, T) \times \mathbb{R}^d \qquad \textbf{(HJ)}$$

$$\frac{\partial m}{\partial t} - div\ (m \nabla u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d \qquad \textbf{(CO)}$$

with the initial and terminal conditions

$$m(0, x) = m_0(x), \quad u(T, x) = G(x, m(T, x)) \quad \text{in } \mathbb{R}^d$$

Fix $\Delta t > 0$, set $K = [\frac{T}{\Delta t}]$ and for $n = 0, 1, ..., K-1$ consider piecewise constant controls

$$\alpha = (\alpha_k)_{k=n}^{K-1} \in \mathbb{R}^{d\times(K-n)}$$

To each α there is an associated discrete dynamics $X_k^{x,n}[\alpha]$ obtained by the recurrence

$$X_n = x\ ; \quad X_{k+1} = X_k - \Delta t\alpha_k = x - \Delta t\sum_{i=n}^{k}\alpha_i \quad \text{for} \quad k = n, ..., K-1$$

A semi-Lagrangian approximation to (HJ)

We describe first the semi-discrete approximation to equation **(HJ)** introduced in CD (1983), see also Ishii-CD (1984)

- the **discrete cost criterion** :

$$J_{\Delta t}(\alpha; x, n) = \Delta t \sum_{k=n}^{K-1} \left[\frac{1}{2} |\alpha_k|^2 + L(k\Delta t, X_k) \right] + G(X_K)$$

- the **discrete value function**:

$$u_{\Delta t}(n, x) = \inf_{(\alpha_k)_{k=n}^{K-1}} J_{\Delta t}(\alpha; x, n) \quad \text{for } k = 0, ..., K-1 \,, \; u_{\Delta t}(K, x) = G(x)$$

- the **discrete (HJ) equation**

$$u_{\Delta t}(n, x) = \inf_{\alpha \in \mathbb{R}^d} \left[u_{\Delta t}(n+1, x - \Delta t\, \alpha) + \frac{1}{2} \Delta t |\alpha|^2 \right] + \Delta t\, L(nh, x)$$

for $n = 1, ..., K-1$ and, for $n = K$, the terminal condition

$$u_{\Delta t}(K, x) = G(x)$$

- **synthesis** : take the argmin in the discrete equation; note that this does not require any regularity at the discrete level and produces suboptimal controls for the original problem

Assume that $L : \mathbb{R}^d \times [0, T] \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ are continuous and

$$\|L(t,.)\|_{C^2} \leq C \quad \forall t \in [0, T], \qquad \|g\|_{C^2} \leq C$$

and set $\hat{u}_{\Delta t}(t,x) = u_{\Delta t}([\frac{t}{h}], x)$. Then,

Theorem

- **uniform semiconcavity**: *$u_{\Delta t}(n, x+y) - 2u_{\Delta t}(n,x) + u_{\Delta t}(n, x-y) \leq C|y|^2$, C independent of h*
- **uniform convergence**: *as $\Delta t \to 0^+$, $\hat{u}_{\Delta t}$ converge locally uniformly in $[0, T] \times \mathbb{R}^d$ to the unique viscosity solution of*

$$-\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = L(x) \quad , \quad u(T,x) = G(x)$$

moreover, $||\hat{u}_{\Delta t} - u|| \leq C\Delta t$

- **regularity**: *$u \in W^{1,\infty}([0, T] \times \mathbb{R}^d)$, u is semiconcave w.r.t x*

Approximation of the continuity equation (CO)

We describe now, following Camilli-Silva (2012), an approximation scheme for the continuity equation :

$$\frac{\partial m}{\partial t} - div\ (m \nabla u) = 0 \quad m(0,x) = m_0(x) \quad \textbf{(CO)}$$

Denote by $\mathcal{P}_1$ the set of probability measures m on $\mathbb{R}^d$ s.t

$$\int_{\mathbb{R}^d} |x| dm(x) < +\infty$$

endowed with Kantorovic-Rubinstein-Wasserstein distance

$$d_1(m_1, m_2) = \sup \left\{ \int_{\mathbb{R}^d} f(x)\, d(m_1 - m_2)(x) \ : f \text{ is -1 Lipschitz} \right\}$$

As a quite subtle consequence of **semiconcavity** of $u_{\Delta t}$, the optimal trajectories for the discrete problem are determined by $\nabla u_{\Delta t}$.
Precisely, the **optimal discrete flow** starting from x is defined by

$$\Phi_0^{\Delta t}(x) = x \, , \; \Phi_{k+1}^{\Delta t}(x) = \Phi_k^{\Delta t}(x) - \Delta t \nabla u_{\Delta t}(k+1, \Phi_k^{\Delta t}(x)), \, k = 1, \ldots, K-1$$

Define now $m_{\Delta t}(k) := \Phi_k[m_0]$ as the **push-forward** of m_0 through the discrete flow, i.e. by asking that, for $k = 1, \ldots, K$,

$$\int_{\mathbb{R}^d} \Psi(x) dm_{\Delta t}(k) = \int_{\mathbb{R}^d} \Psi(\Phi_k^{\Delta t}(x)) m_0(x) \, dx$$

for any $\Psi \in C(\mathbb{R}^d)$.

Theorem

As $\Delta t \to 0^+$, the discrete measures $m_{\Delta t}$ converge to a measure m in $C([0, T]; \mathcal{P}_1)$ which solves **(CO)** *in the sense of distributions.*

The proof uses, among other, the following estimates:

- $|\Phi_k^{\Delta t}(x) - \Phi_k^{\Delta t}(y)|^2 \geq (\frac{1}{1+C_1+C_2\Delta t})^k |x-y|^2$
- $d_1(m_{\Delta t}(k_1), m_{\Delta t}(k_2)) \leq C\Delta t|k_1 - k_2|$
- $m_{\Delta t}(k)$ absolutely continuous, bounded support independent of k

The semi-discrete scheme for the MFG system

The complete semi-discrete scheme is

$$u_{\Delta t}(k,x) = \inf_{\alpha\in\mathbb{R}^d}\left[u_{\Delta t}(k+1, x-\Delta t\,\alpha)+\frac{1}{2}\Delta t|\alpha|^2\right]+\Delta t\, L(x, m_h(k))\ , n = 1, ..., K$$

$$m_{\Delta t}(k) = \Phi_k^{\Delta t}[m_0] \quad , m_{\Delta t}(0) = m_0 \in \mathcal{P}_1$$

$$u_{\Delta t}(K,x) = G(x, m_{\Delta t}(K))$$

Remember that the flow $\Phi_k^{\Delta t}[m_0]$ is constructed via the optimization procedure dictated by the solution of **discrete (HJ)**

The following well-posedness result due to Camilli-Silva (2012) holds:

Theorem

For sufficiently small time step Δt*:*

- *the discrete system has a solution* $(u_{\Delta t}, m_{\Delta t}) \in C([0,T] \times \mathbb{R}^d) \times C([0,T]; \mathcal{P}_1)$

If, in addition, for all $m_1, m_2 \in \mathcal{P}_1, m_1 \neq m_2$

- $\int_{\mathbb{R}^d} (L(x, m_1) - L(x, m_2))\, d(m_1 - m_2)(x) > 0$
- $\int_{\mathbb{R}^d} (G(x, m_1) - G(x, m_2))\, d(m_1 - m_2)(x) \geq 0$

then the solution is unique.

As $\Delta t \to 0$*:*

- $u_{\Delta t}$ *converges to* u *locally uniformly to* u*,*
- $m_{\Delta t}$ *converges to* m *in* $C([0,T]; \mathcal{P}_1)$*,*

where (u, m) *is the unique solution of system* **MFG**

The proof of existence for the discrete system makes use of a fixed point argument for a suitably defined map S on the space of measures.

The necessary continuity of S follows in particular from the following compactness property of sequences of semiconcave functions, see Cannarsa -Sinestrari:

Lemma

Suppose u_k are uniformly semiconcave and uniformly bounded. Then at least a subsequence u_{k_j} converge locally uniformly to a semiconcave function u and, moreover, ∇u_{k_j} converge a.e. to ∇u