Around the Cauchy-Kowalewski theorem

Sergei Kuksin

(based on a joint work with Nikolai Nadirashvili

Cortona, 21 May 2012

§1. Introduction

Consider the Cauchy problem for a quasilinear wave equation:

(1) $$\Box u + f(t,x,u,\nabla u,\dot u) = 0, \qquad \dim x = d; \quad t \geq 0;$$

(2) $$u_{t=0} = u_0, \qquad \dot u_{t=0} = u_1.$$

We can study it globally, when

x belongs to a compact Riemann manifold M and $\Box u = \ddot u - \Delta u$, where Δ – Laplace - Beltrami oparator.

Or locally, when

x belongs to a characteristic cone for $\Box u = \ddot u - \Delta u$, where Δ – Laplace - Beltrami oparator. w.r.t. some Riemann metric on $\mathbb{R}^d$

To begin I will assume that

$$x \in \mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d.$$

Let f and u_0, u_1 be analytic in all its arguments. What do we know about the solutions?

I) The Cauchy-Kowalewski theorem: There exists $\varepsilon_1 > 0$ such that for $0 \leq t \leq \varepsilon_1$ and $x \in \mathbb{T}^d$ problem (1), (2) has a unique analytic solution.

S. V. Kowalewski, *Zur Theorie der partiellen Differentialgleichungen*, J. Reine Angew. Math. 80 (1875).

In the proof ε_1 is the radius of analyticity, so this is a small number. What happens for $t > \varepsilon_1$?

II) The Ovsiannikov-Nirenberg theorem: Let f be continuous in t and analytic in all other variables, as well as u_0 and u_1. Then there exists $\varepsilon_2 > 0$ such that for $0 \leq t \leq \varepsilon_2$ and $x \in \mathbb{T}^d$ problem (1), (2) has a unique solution, analytic in x and C^1 in t.

T. Nishida, *A note on a theorem of Nirenberg*, J. Diff. Geom. 12 (1977).

It is also known that

III) Theorem: If f is sufficiently smooth in x, then there exists $T > 0$ such that for $0 \leq t \leq T$ and $x \in \mathbb{T}^d$ the problem has a classical solution.

Sometimes T is fairly large, e.g. $T = \infty$.

If both theorems I) and III) apply, then, clearly, $T \geq \varepsilon_1$. But is $T > \varepsilon_1$, or $T = \varepsilon_1$?

§2. Main Result.

Choose $m > d/2$, assume that the nonlinearity $f(t, x, u, \nabla u, \dot{u})$ is continuous in t, H^m-smooth in x and analytic in $u, \nabla u, \dot{u}$. Let $u_0 \in H^m, u_1 \in H^{m+1}$.

Main Theorem (propagation of analyticity): Let $u(t, x),\ 0 \le t \le T, x \in \mathbb{T}^d$, be a solution of the Cauchy problem (1), (2) which is H^{m+1}-smooth in x. Then:

i) If u_0, u_1 and f are real-analytic in $(x_1, \dots, x_k), 1 \le k \le d$, then u also is analytic in these variables.

ii) If u_0, u_1 and f are real-analytic in all their arguments, then u also is.

Assertion ii) was known, see:

S. Alihnac and G. Metivier, *Propagation de l'analyticité ...*, Invent. Math. 75 (1984).

The proof of this work uses heavy tools of paradifferential calculus (and their result applies to strongly nonlinear hyperbolic equations). Also see

C. Bardos and S. Benachour, *Domaine d 'analyticite des solutions de l'equation d'Euler dans un ouvert de R^n.* Ann. Scu. Norm. di Pisa, 4(1977) where similar result is obtained for solutions of the Euler equation (using its hyperbolic features).

Assertion i) and Theorem III) (on local in time existence of a classical solution) imply a generalisation of the Ovsiannikov-Nirenberg theorem.

§3. Discussion of the proof.

I am speaking about the Cauchy problem (1)-(2):

$$\Box u + f(t, x, u, \nabla u, \dot{u}) = 0, \qquad u(0, x) = u_0(x), \;\; \dot{u}(0, x) = u_1(x).$$

Denote $\mathcal{H}^m = H^{m+1} \times H^m$, $m > d/2$. This will be the space of Cauchy data: $(u_0, u_1) \in \mathcal{H}^m$. Consider the Cauchy operator for the linear wave equation:

$$\widetilde{\Box} : u \mapsto (u_{t=0}, \dot{u}_{t=0}, \Box u).$$

This is an embedding. For any $T > 0$ consider the spaces

$$X_m^T = C(0, T; H^{m+1}) \cap C^1(0, T; H^m), \qquad Y_m^T = \mathcal{H}^m \times C(0, T; H^m).$$

X_m^T - space of solutions, Y_m^T - space of Cauchy data and the r.h.s.'s.
I will call solutions $u(t, x) \in X_m^T$ classical solutions. We have

$$\widetilde{\Box}^{-1} : Y_m^T \to X_m^T, \quad \text{but} \quad \widetilde{\Box} : X_m^T \to \mathcal{H}^m \times C(0, T; H^{m-1}).$$

In this scales of function spaces we lose 1 unit of smoothness, when apply $\widetilde{\Box}$ after $\widetilde{\Box}^{-1}$. Not good! Let us use a simple trick:

Denote $\widetilde{\Box}^{-1}Y_m^T = Z_m^T$ and provide the space Z_m^T with a norm, induces from Y_m^T. This is a Banach space such that

$$1)\ \ \widetilde{\Box} : Z_m^T \to Y_m^T \qquad \text{is an isomorphism,}$$

$$2)\ \ Z_m^T \subset X_m^T \qquad \text{continuously,} \quad \text{and} \ \ X_{m+1}^T \subset Z_m^T,$$

since $\widetilde{\Box}^{-1} : Y_m^T \to X_m^T$. Denote by Φ the operator of nonlinear Cauchy problem:

$$\Phi(u) = (u_{t=0}, \dot{u}_{t=0}, \Box u + f(t, x, u, \nabla u, \dot{u})).$$

Since $m > d/2$, then the space $C(0, T; H^m)$ is a Banach algebra. As $u, \nabla u, \dot{u} \in C(0, T; H^m)$, then the mapping

$$\Phi : Z_m^T \to Y_m^T \qquad \text{is analytic.}$$

Problem (1), (2) with 0 in the r.h.s. replaced by a function $g(t,x)$ has a unique solution. So

$$\Phi : Z_m^T \to Y_m^T \qquad \text{is an analytic embedding.}$$

Consider differential of Φ at any point $u \in Z_m^T$:

$$d\Phi(u)(v) = \big(v_{t=0}, \dot v_{t=0}, \Box v + d_3 f[u]v + d_4 f[u]\nabla v + d_5 f[u]\dot v\big).$$

Here $f[u] = f(x, u, \nabla u, \dot u)$. Easy to see that this also is an isomorphism $Z_m^T \overset{\rightarrow}{\sim} Y_m^T$. Since Φ is an embedding, then the inverse function theorem implies

Lemma: Φ is an analytic diffeomorphism of the space Z_m^T and a domain $\mathcal{O} \subset Y_m^T$, $\Phi : Z_m^T \overset{\rightarrow}{\sim} \mathcal{O}$.

Denote

$$\mathcal{O}^0 = \{(u_0, u_1) \in \mathcal{H}^m \mid (u_0, u_1, 0) \in \mathcal{O}\}$$

Then for $0 \le t \le T$ the flow-maps

$$S_0^t : \mathcal{O}^0 \to \mathcal{H}^m, \qquad (u_0, u_1) \to (u(t), \dot u(t)),$$

are well defined and analytic.

So: There is a domain $\mathcal{O}^0 = \mathcal{O}^0([0,T]) \subset \mathcal{H}^m$ such that the problem

$$\Box u + f(t,x,u,\nabla u,\dot{u}) = 0,\ 0 \le t \le T, \qquad u(0,x) = u_0(x),\ \ \dot{u}(0,x) = u_1(x).$$

has a classical solution $u \in X_m^T$ iff $(u_0,u_1) \in \mathcal{O}^0$. This solution analytically depends on (u_0,u_1). If f analytically depends on some extra parameter ξ, then $\mathcal{O}^0 = \mathcal{O}^0_\xi$ and the solution u also analytically depends on ξ.

Introducing the parameters.

For simplicity let $k = d$. Then u_0, u_1 and f are analytic in x. The space $\mathbb{R}^d = \{\theta = (\theta_1, \dots, \theta_d)\}$ acts on $\mathbb{T}^d$ by the shifts ${}_\theta R$,

$$ {}_\theta R(x) = (x + \theta) $$

Accordingly it acts on the nonlinear operators $f(t, x, u, \nabla u, \dot{u})$ by shifting their coefficients:
$$\big({}_\theta R f\big)(t, x, u, \nabla u, \dot{u}) = f(t, {}_\theta R x, u, \nabla u, \dot{u}).$$
Clearly we have

$$(\Box + {}_\theta R f(t, x, u, \nabla u, \dot{u}))({}_\theta R u) = {}_\theta R\big((\Box u + f(t, x, u, \nabla u, \dot{u}))\big).$$

Consider operator of the shifted Cauchy problem ${}_\theta\Phi(u) = (u_{t=0}, \dot{u}_{t=0}, \Box u + {}_\theta R f(u)$.
It defines an analytic mapping

$$\bar{\Phi} : \mathbb{T}^k \times Z_m^T \to Y_m^T, \qquad (\theta, u) \to {}_\theta\Phi(u).$$

For any θ denote by ${}_\theta u(t)$ solutions of the shifted equation $\Box + {}_\theta R f$, and by ${}_\theta S_0^t$, $0 \le t \le T$, flow-maps of that equation.

Consider $\bar{\Phi}$ for θ in a small ball $B_\varepsilon = \{|\theta| \le \varepsilon\}$. If $(u_0, u_1) \in \mathcal{O}^0$, then $(u_0, u_1, 0) \in Y_m^T$ is a regular value for $\bar{\Phi}(0, \cdot)$. By the Implicit Function Theorem, for $\theta \in B_\varepsilon$ and (u'_0, u'_1) close to (u_0, u_1) the flow-maps for the shifted equation ${}_\theta S_0^t : (u'_0, u'_1) \mapsto ({}_\theta u(t), {}_\theta \dot{u}(t)),\ 0 \le t \le T,$ are well defined, analytic in θ and in (u'_0, u'_1) . We have:

$$\boxed{{}_\theta S_0^t \circ {}_\theta R(u_0, u_1) = {}_\theta R \circ S_0^t(u_0, u_1),} \qquad \text{if } \theta \in B_\varepsilon.$$

Consider a solution of the Cauchy problem (1), (2), $u(t, x) = S_0^t(u_0, u_1)$. The term on the right is

$$ {}_\theta R \circ S_0^t(u_0, u_1) = u(t, x + \theta), $$

and the term on the left is

$$ {}_\theta S_0^t \circ {}_\theta R(u_0, u_1) = {}_\theta S_0^t\Big((u_0, u_1)(x + \theta)\Big), \qquad \theta \in B_\varepsilon. $$

We assumed that u_0 and u_1 are analytic. Then $(u_0, u_1)(x + \theta)$ is analytic in θ. As the operator ${}_\theta S_0^t(u'_0, u'_1)$ is analytic in $\theta \in B_\varepsilon$, then ${}_\theta S_0^t\big((u_0, u_1)(x + \theta)\big)$ also is. So $u(t, x + \theta)$ is analytic in $\theta \in B_\varepsilon$!

We have proved the first assertion of the Main Theorem:

i) If u_0, u_1 and f are real-analytic in $(x_1, \ldots, x_k)$, $1 \leq k \leq d$, then u also is analytic in these variables.

To prove the second assertion we have to show that u is analytic in t. By analogy, we have to shift the Cauchy data u_0, u_1 not in x-variable, but in t-variable. How to do this? – Apply the Cauchy-Kowalevski theorem to find the solution $u(\theta, x)$, $|\theta| < \varepsilon$! It is analytic, so it gives the needed time-θ shifts of the Cauchy data, analytic in θ. Now we argue as before to prove that $u(t, x)$ is analytic in t, till it exist as a classical solution. We assumed that the Cauchy-Kowalevski theorem is applicable, i.e. that all the data are analytic.

$\S$4. Related results.

i) *Equations in homogeneous spaces.*

The proof applies to quasilinear wave equations in a compact Riemann homogeneous space. In this case $\Box = \partial^2/\partial t^2 - \Delta$, where Δ is the corresponding Laplace-Beltrami operator. Now the translations ${}_\theta R$ should be replaced by the local isometies. For example, the theorem remans true for quasilinear wave equations on the standard sphere S^d.

ii) *A local version of the result.* Consider the problem

$$\Box u + f(t, x, u, \nabla u, \dot{u}) = 0,\ 0 \le t \le T, \qquad u(0, x) = u_0(x),\ \ \dot{u}(0, x) = u_1(x).$$

in the characteristic cone

$$\{(t, x) \in [0, T] \times \mathbb{R}^d : |x| < T - t\},$$

when the Cauchy data are given at the ball $\{|x| < T\}$. Then a natural version of the results hold true with the same proof.

iii) *Quasilinear parabolic equations.*

The approach to study analyticity and partial analyticity of solutions in x (but not in t) applies to other equations. For example, to quasilinear parabolic equations

(3) $$\dot{u} - \Delta u + f(t, x, u, \nabla u) = 0, \quad x \in \mathbb{T}^d, \;\; t \geq 0, \qquad u_{t=0} = u_0,$$

where f is sufficiently smooth in t, x and is analytic in u and ∇u. One can find suitable space Z_m^T and Y_m^T such that the operator Φ of the Cauchy problem (3) defines an analytic diffeomorphism between Z_m^T and a subdomain of the space Y_m^T, see
[SK82], *Diffeomorphisms of functional spaces that correspond to quasilinear parabolic equations*, Math. USSR Sbornik 117 (1982).

In the same way as before we prove that if f is analytical in all its variables, then classical solutions of (3) with analytical initial data are analytical in x. This is a well known result. But we also can prove that if f is analytic in $u, \nabla u$ and in a part of the space-variables, as well as the function u_0, then the solution $u(t, x)$ as well is analytic in these space variables. This result seems new.

This approach applies to the Navier-Stokes system on the d-torus with $d = 2$ or $d = 3$, perturbed by a sufficiently smooth force $h(t, x)$, see [SK82]. It implies that if the initial data and the force h are analytical in space-variables $x_1, \dots, x_k$, where $1 \le k \le d$, then a corresponding strong solution $u(t, x)$ remains analytic in this space-variables till it exists.

Example. Consider the 3d NSE in the spherical layer $S^2 \times (0, \varepsilon) = \{(\varphi, r\})$. Let the force and initial data are

i) analytic in φ,

ii) bounded uniformly in $t \ge 0$, uniformly in $\varepsilon \in (0, 1)$.

Due to Raugel-Sell, if positive ε is sufficiently small, then there exists a unique strong solution $u(t, \varphi, r),\ t \ge 0$. By our results this solution is analytic in φ.

iv) *NLS equation*

The result remains true for the nonlinear Schrödinger equation

$$\dot{u} - i\Delta u + f(t, x, \mathrm{Re}\, u, \mathrm{Im}\, u) = 0, \qquad u_{t=0} = u_0, \quad x \in \mathbb{T}^d. \tag{4}$$

Function $f \in \mathbb{C}$ is continuous in t, H^m-smooth in x ($m > d/2$), analytic in $\mathrm{Re}\, u, \mathrm{Im}\, u$. We can replace $\mathbb{T}^d$ by any homogeneous Riemann space, analytic and compact.

§5. Energy Transfer to High Frequencies.

Consider again eq. (4), where u_0 is analytic, f is continuous in t, analytic in x, Re u, Im u:

$$\dot u - i\Delta u + f(t,x,\text{Re}\, u, \text{Im}\, u) = 0, \qquad u_{t=0} = u_0, \quad x \in \mathbb{T}^d.$$

Solution $u(t,x)$ is analytic in x, continuous in t. Denote

$\rho(t) =$ radius of analyticity of $u(t,x)$ in $x = \min\left\{\text{Im}\, z \mid z-\text{ singular point of } u(t,z)\right\}$.

Set

$$C(t) = \sup\left\{|u(t,z)| \mid |\text{Im}\, z| \le \tfrac{1}{2}\rho(t)\right\}$$

and write

$$u(t,x) = \sum_s u_s(t) e^{is\cdot x}.$$

Then

$$|u_s(t)| \le C(t) e^{-\frac{1}{2}\rho(t)|s|} \quad \forall\, s,$$

$$\rho(t) = -\limsup_{s\to\infty} \big(\ln |u_s(t)| \big) \, |s|^{-1}.$$

So,

eq. (4) exhibits the energy transfer to high modes iff $\liminf_{t\to\infty} \rho(t) = 0$.

Example. Let (4) be the 1d defocusing Zakharov-Shabat equation. Its solutions $u(t,x)$ are given by the Its-Matveev-McKean-Trubowitz formula. Therefore each $u(t,x)$ is a meromorphic function of x and

$$\rho(t) = \min\{|\text{Im} z| \mid z - \text{pole of } u(t,x) \text{ in } x\}.$$

Now the function $\rho(t)$ is almost-periodic and $\rho \ge \rho_0 > 0$.

No energy transfer!

Example. Consider

$$(5) \qquad \dot{u} - \delta i \Delta u + i|u|^{2p} u = 0, \qquad u_{t=0} = u_0, \quad x \in \mathbb{T}^d,$$

where $\delta > 0$, $u_0(x)$ is analytic and $|u_0| \sim L$. Low estimates on Sobolev norms of $u(t)$ from

SK, GAFA 9 (1999), 141-184

imply that

$$\liminf_{t\to\infty} \rho(t) \le C\Big(\frac{\delta}{L^{2p}}\Big)^{1/3}.$$

Conjecture. If in (5) $d \ge 2$, then for a typical solution u we have $\liminf_{t\to\infty} \rho(t) = 0$.

iv) *Strongly nonlinear equations.*

Consider a “strongly quasilinear” wave equation:

$$\ddot{u} + \frac{\partial}{\partial x_j} A_{jk}(t, x, u, \nabla u, \dot{u}) \frac{\partial u}{\partial x_k} + f(t, x, u, \nabla u, \dot{u}) = 0,$$

The Cauchy-Kowalewski theorem still applies to the corresponding Cauchy problem. Does the principe of propagation of analyticity holds? Yes it does, but some extra ideas have to be used for a proof.

REFERENCE

SK, N. Nadirashvili “Analyticity of solutions for quasilinear wave equations”, preprint 2012, 12 p., (arXiv 1205.5926)