On the Role of Fields of Abnormal extremals in Geometry of Distributions

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By complete analogy one can define a local version of this equivalence relation considering the action of germs of diffeomorphisms on germs of (ℓ, n)-distributions.

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The tuple $(\dim D(q), \dim D^2(q), \ldots, \dim D^j(q), \ldots)$ is called the *small growth vector of D at the point q* (or, shortly, s.v.g.).

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   $(3, 6)$-distribution with s.g.v. $(3, 6)$ R. Bryant, 1979

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Question: What is the algebra of infinitesimal symmetries of the flat distribution of type $\mathfrak{m}$?

Universal algebraic prolongation & symmetries of the flat distribution

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Its calculation is reduced to pure Linear Algebra

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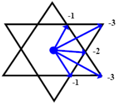

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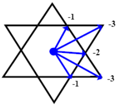

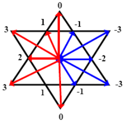

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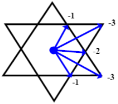

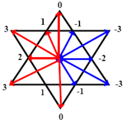

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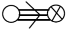

The grading corresponds to the marking of the shorter root in the Dynkin diagram of G_2.

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Theorem (Tanaka, 1970)

1. *To a distribution D with constant symbol $\mathfrak{m}$ one can assign in a canonical way a bundle over M of dimension equal to $\dim \mathfrak{U}(\mathfrak{m})$ equipped with a canonical frame.*

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3. *This upper bound is sharp and is achieved if and only of a distribution is locally equivalent to the flat distribution $D_{\mathfrak{m}}$.*

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Therefore *Tanaka's approach allows one to predict the number of prolongations steps and the dimension of the bundle, where the canonical frame lives,* **without making concrete normalizations on each step** (as the original Cartan method of equivalence suggests)

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In order to apply this machinery to all bracket-generating (ℓ, n)-distributions with fixed ℓ and n, one has

1. to classify all n-dimensional graded nilpotent Lie algebras with ℓ generators.- hopeless task in general;
2. to generilize the Tanaka prolongation procedure to distributions with nonconstant symbol, because the set of all possible symbols may contain moduli.

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- Moduli appears for symbols of $(2,n)$ distributions starting from $n = 8$.

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The origin of the method - **Optimal Control Theory**

The key idea: study of the flow of abnormal extremals

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Abnormal extremals do not depend on the functional but on the distribution D only.

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- If $\mathrm{rank} D = 2$, then $\mathcal{H}_D = (D^2)^\perp$

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Abnormal extremals (continued) and the lift of D to $\mathcal{H}_D$

Let C be the Cauchy characteristic distribution of $\widetilde{\Delta}$, i.e. a subdistribution of $\widetilde{\Delta}$ such that $[C, \widetilde{\Delta}] \subset \widetilde{\Delta}$.

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We work with the distributions C, V, and J instead of the original distribution D.

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Let γ be a segment of an abnormal extremal, O_γ be a neighborhood of γ in $\mathcal{H}_D$ s.t. the factor

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The curve $\lambda \to F_\gamma(\lambda), \lambda \in \gamma$ is a curve of coisotropic subspaces of $\Delta(\gamma) \subset T_\gamma N$, called the Jacobi curve of the abnormal extremals γ.

The role of Jacobi curves

- Any invariant of the Jacobi curve F_γ w.r.t the action of (Conformal) Symplectic Group on the corresponding Grassmannian of coisotropic subspaces (or, shortly, symplectic flags) of $\Delta(\gamma)$ produces an invariant of the distribution D.

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the canonical frame for D itself on certain fiber bundle over $\mathcal{H}_D$

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- The fundamental form of Jacobi curves of abnormal extremals gives the Cartan invariant of $(2,5)$-distributions and therefore generalize it to $(2,n)$-distributions for arbitrary $n > 5$ (Zelenko, 2004)

Sketch of developments (continued) and some obstacles

- Geometry of Jacobi curves of rank 2 distributions can be reduced to the geometry of so-called self-dual curves in a projective space. Using this fact and existence of the canonical projective structure of item 1, one can construct the canonical frame for $(2, n)$-distributions for arbitrary $n > 5$ (Boris Doubrov and Zelenko, 2005)

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It cannot be reduced in general to geometry of curves in a Lagrangian Grassmannian or curves in projective spaces.
"Naive", by hand constructions of canonical moving frames for such curves might be very cumbersome (were implemented by Doubrov and Zelenko, 2008 in the case of $(3, n)$-distributions for arbitrary $n > 5$)

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$$F_\gamma^i(\lambda) := \begin{cases} (F_\gamma^{-i-1}(\lambda))^\angle & \text{if } \; F_\gamma^{-1}(\lambda) \;\; \text{is proper coisotropic} \\ (F_\gamma^{-i-2}(\lambda))^\angle & \text{if } \; F_\gamma^{-1}(\lambda) \;\; \text{is Lagrangian} \end{cases}$$

i.e $F_\gamma(\lambda)$ is a symplectic flag for any $\lambda \in \gamma$;

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The tangent vector to the Jacobi curve at a point corresponding to λ can be identified with a line $s_\lambda \subset \mathfrak{csp}\left(\oplus_{\mathrm{i}\in\mathbb{Z}}\mathrm{Gr}^{\mathrm{i}}(\lambda)\right)$ of degree -1, i.e. s.t. $s_\lambda(\mathrm{Gr}^i(\lambda)) \subset \mathrm{Gr}^{i-1}(\lambda)$

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s_λ is called the symbol of the Jacobi curve at λ.

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For fixed rank D and $\dim M$ **the set of all possible symbols of Jacobi curves, up to an isomorphism, is finite**.

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Actually this follows from more general fact (E.Vinberg, 1976): *If G is a semisimple Lie group, $\mathfrak{g}$ is its Lie algebra with given grading $\mathfrak{g} = \oplus_{i=-\mu}^{\mu} \mathfrak{g}_i$, and G_0 is the connected subgroup of G with the Lie algebra $\mathfrak{g}_0$, then the set of orbits of elements of $\mathfrak{g}_{-1}$ w.r.t. the adjoint action of G_0 is finite.*

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Finiteness of the set of symbols, up to isomorphism+ classification of symplectic symbols

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$$\underbrace{s}_{\text{Jacobi symbol of the distribution } D \text{ at } q} \subset \mathfrak{csp}_{-1}(\underbrace{\oplus X^i}_{\text{fixed graded symplectic space } V := \oplus X^i})$$

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Theorem (Doubrov-Zelenko) *To a curve of flags of isotropic/coisotropic subspaces with constant symbol* s *one can assign in a canonical way a bundle of moving frames of dimension equal to* $\dim \mathfrak{U}_F(s)$.

Distributions of maximal class

Jacobi curve of a generic abnormal extremal γ satisfies $F_\gamma^{-i(\lambda)}(\lambda) = \Delta(\gamma)$ for some integer $i(\lambda)$

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For example, all $(2, 6)$-distributions with hyperbolic, parabolic, and elliptic Tanaka symbols have the same Jacobi symbol.

From canonical moving frames for Jacobi curves to canonical frames for distributions

Build the following graded Lie Algebra

$$B(s) = \underbrace{\overbrace{\eta}^{g^{-2}} \oplus \overbrace{\underbrace{(\oplus X^I)}_{V}}^{g^{-1}}} \oplus \overbrace{\mathfrak{U}_F(s)}^{g^0}$$

The Heisenberg algebra - the Tanaka symbol of the contact distributionΔ

Main Theorem on distributions with given Jacobi symbol

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Moreover, if in addition $\mathrm{rank} D = 2$ or $\mathrm{rank} D$ is odd, this upper bound for the algebra of infinitesimal symmetries is sharp.

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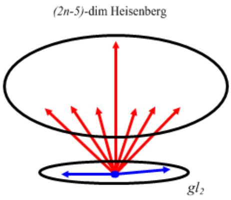

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A distribution is called controllable by abnormal trajectories, if any two points can be connected by a concatenation of abnormal trajectories $\Leftrightarrow$ the distribution $V \oplus C$ is bracket-generating.

Theorem (**Doubrov-Zelenko**) If a distribution D is controllable by abnormal trajectories, then it has a finite dimensional algebra of infinitesimal symmetries

THANK YOU FOR YOUR ATTENTION