Example of endowed Lie algebras and their prolongations in the classification of $(2,5)$ distributions (the problem in E. Cartan's "5 Variables" paper)

Theorem. Let $\mathcal{E} = (\mathcal{A},\ P, (b))$ be a 6-dim Lie algebra endowed with a generating 2-plane and 1-dimensional isotropy subalgebra (b). If $\mathcal{E}$ is not a prolongation of the graded nilpotent 5-dim Lie algebra then the linear operator $ad\ b|_P$ has zero trace. If it is non-singular then the endowed algebra is isomorphic (over $\mathbb{C}$) to the endowed Lie algebra

$$[a_1, a_2] = a_3, \quad [a_1, a_3] = a_4, \quad [a_2, a_3] = a_5, \qquad [a_1, b] = a_2, \quad [a_2, b] = -a_1$$

$$[a_1, a_4] = [a_2, a_5] = \lambda_1 a_3 + \lambda_2 b, \qquad [a_1, a_5] = [a_2, a_4] = 0$$

(the other brackets are defined by Jacobi identity), endowed with the plane (a_1, a_2) and isotropy subalgebra (b). At least one of λ_1, λ_2 is non-zero, and

$$\lambda = \lambda_1^2/\lambda_2$$

is a complete invariant with respect to isomorphisms of endowed algebras. As a non-endowed algebra it is $so(3) \oplus so(3)$ provided $\lambda_2 \neq 0$ and $\lambda \neq 4$. For such λ, the endowed algebra $\mathcal{E}$ has the following properties:

- it is a prolongation of the 5-dimensional Lie algebra

$$[X_1, X_2] = X_3, \ [X_2, X_3] = X_1, \ [X_3, X_1] = X_2, \quad [Y_1, Y_2] = Y_2, \quad [X_i, Y_j] = 0$$

endowed with generating 2-plane spanned by $a_1 = Y_1 + \theta X_1, \quad a_2 = Y_2 + X_2$

- it describes the problem of rolling balls of radii R_1, R_2.

The relation between $\lambda,\ \theta$, and R_1, R_2 is as follows:

$$\lambda = -\left(\frac{1-\theta^2}{\theta}\right)^2 = \left(\frac{1+\mu^2}{\mu}\right)^2, \quad where \ \ \mu = \frac{R_1}{R_2}.$$

Theorem. The endowed Lie algebra $\mathcal{E}$ is prolongable if and only if

$$\lambda = 100/9 \quad \Leftrightarrow \quad \theta = \pm 3i \quad \Leftrightarrow \quad \mu = R_1/R_2 \in \{3,\ 1/3\}$$

and in this case the prolongation is g_2.

Remark. The fact that in the problem of rolling balls the symmetries form g_2, on top of "visible" $so(3) \oplus so(3)$, if and only if the ratio of the radii is $1:3$ is R.Bryant's theorem (lecture notes, 1998). He proved it by computing Cartan tensor, following Cartan's constructions. First published proof is due to I. Zelenko (2006); it is also computation of Cartan tensor, but with a different construction based on his joint approach with A. Agrachev. Another published proof, based on a work with Lie algebras, is due to G. Bor and R. Montgomery (2009). A nice math related to rolling balls (split-octonions) is contained in one of A. Agrachev's works (2007).

Homogeneous bracket generating affine distribution $A+(B)$ of $\mathbb{R}^3$ and $\mathbb{R}^4$

Theorem. The symmetry algebra of a homogeneous affine distribution germ $D = A + (B)$ on $\mathbb{R}^n$, $n = 3, 4$, is one of the following:

(a) $n = 3, 4$: any n-dimensional Lie algebra admitting a generating 2-plane

(b) $n = 4$: one of the 5-dimensional solvable Lie algebras

$$\begin{array}{lllll} & [b,a]=c & [a,c]=\theta b & [w,b]=b & [w,a]=0 \\ \mathcal{A}^5_\theta: & [b,c]=d & [a,d]=0 & [w,c]=c & [w,d]=2d \\ \theta\in\{1,0\} & [b,d]=0 & [c,d]=0 & & \end{array}$$

(c) $n = 3, 4$: Lie algebra of vector fields germs on $\mathbb{R}^2$ of the form $c\frac{\partial}{\partial y_1} + f(y_1, y_2)\frac{\partial}{\partial y_2}$, $c \in \mathbb{R}$.

(d) $n = 3, 4$: Lie algebra of vector field germs on $\mathbb{R}$

In each of these cases **except case (d) for $\mathbf{n} = \mathbf{4}$** the affine distribution D can be described by an 4-dimensional Lie algebra endowed with an affine line.

Theorem. The symmetry algebra is ∞-dimensional (one of the cases cases **(c)**, **(d)**) if and only if

$$[B, [B, A]] \in (B, [B, A]]) \,.$$

In each of the cases **(c)** and **(d)** all affine distributions are diffeomorphic. These cases hold if and only if

	$\mathbb{R}^3$	$\mathbb{R}^4$
Case (c)	$[B,[B,A]] \in (B,[B,A]])$ $[A,[B,A]] \in (B,[B,A]])$ Normal form: $B = \frac{\partial}{\partial x_1}$ $A = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}$	$[B,[B,A]] \in (B,[B,A]])$ $[A,[A,[B,A]] \in (B,[B,A]],[A,[B,A]])$ Normal form: $B = \frac{\partial}{\partial x_1}$ $A = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}$
Case (d)	$[B,[B,A]] \in (B,[B,A]])$ $[A,[B,A]] \notin (B,[B,A]])$ Normal form: $B = \frac{\partial}{\partial x_1}$ $A = (1 + x_1x_2)\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}$	$[B,[B,A]] \in (B,[B,A]])$ $[A,[A,[B,A]] \notin (B,[B,A]],[A,[B,A]])$ Normal form: $B = \frac{\partial}{\partial x_1}$ $A = (1 + x_2x_3)\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}$

Remark. The conditions in the table imply that $D = A+(B)$ is homogeneous for $\mathbb{R}^3$ and case (**c**) for $\mathbb{R}^4$. In the case (**d**) for $\mathbb{R}^4$ the affine distribution is homogeneous if and only if it satisfies the given conditions and the condition

$$[B, [A, [A, [B, A]] \in (B, [B, A]], [A, [B, A]]) \,.$$

3-dim Lie algebras $\mathcal{A}$ admitting a generating line $a+(b)$ and distinguishing prolongable endowed algebras

3-dim Lie algebra $\mathcal{A}$ admitting a generating $a+(b)$	# moduli in classif. of $\mathcal{A}$ and classif. of endowed $\mathcal{A}$	endowed algebra is prolongable if and only if	Prolongation (if prolongable)
$so(3)$	0; 1	never prolongable	
$sl(2)$	0; 1	$dim(b,[b,\mathcal{A}])=2$	$\mathcal{A}_{\infty,1}$
$(3,2,0)$ $dim\mathcal{A}^2=2$ $\left(\mathcal{A}^2\right)^2=0$	1; $b\notin\mathcal{A}^2$: 1 $b\in\mathcal{A}^2$: 2	$b\in\mathcal{A}^2$	$\mathcal{A}_{\infty,2}$
$(3,1,0)$ non-nilp. $dim\mathcal{A}^2=1$ $[\mathcal{A},\mathcal{A}^2]\neq 0$	0; 1	always prolongable	$\mathcal{A}_{\infty,2}$
$(3,1,0)$ nilp. (Heisenberg) $dim\mathcal{A}^2=1$ $[\mathcal{A},\mathcal{A}^2]=0$	0; 0	always prolongable	$\mathcal{A}_{\infty,2}$

$\mathcal{A}_{\infty,1}$: Lie algebra of vector field germs on $\mathbb{R}$

$\mathcal{A}_{\infty,2}$: Lie algebra of vector fields of the form $c\frac{\partial}{\partial y_1}+f(y_1,y_2)\frac{\partial}{\partial y_2}$, $c\in\mathbb{R}$.

Classification of homogeneous affine distributions $A + (B)$ on $\mathbb{R}^3$ = classification of non-prolongable 3-dim Lie algebras endowed with generating affine line $a + (b)$

3-dim Lie algebra $\mathcal{A} = (a, b, c)$	normal form for $\mathcal{A}$ endowed with $a + (b)$	parameters
$so(3)$	$[b, a] = c \quad [a, c] = tb \quad [c, b] = a$	$t > 0$
$sl(2)$	$[b, a] = c \quad [a, c] = tb \quad [c, b] = a$	$t > 0$
	$[b, a] = c \quad [a, c] = tb \quad [c, b] = -a$	$t \neq 0$
$(3, 2, 0)$ $dim\mathcal{A}^2 = 2$ $\left(\mathcal{A}^2\right)^2 = 0$	$[b, a] = c \quad [b, c] = \pm a + \lambda c \quad [a, c] = 0$	$\lambda \geq 0$, $\pm$: invariants of $\mathcal{A}$

Normal forms:

$so(3)$: take any vector fields X_1, X_2, X_3 on $\mathbb{R}^3$, independent at 0, such that $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$. A normal form for affine distribution is $A + (B)$, $B = X_1$, $A = \sqrt{t}X_2$.

$sl(2)$: similar

$(4, 2, 0)$: take any vector fields A, B, C on $\mathbb{R}^3$, independent at 0, such that $[B, A] = C$, $[B, C] = \pm A + \lambda C$, $[A, C] = 0$. A normal form is $A + (B)$, for example $A = \frac{\partial}{\partial x_2}$, $B = \frac{\partial}{\partial x_1} \mp x_3 \frac{\partial}{\partial x_2} + (x_2 + \lambda x_3) \frac{\partial}{\partial x_3}$, $\theta \geq 0$.

Prolongable 3-dim Lie algebras endowed with generating affine line $a + (b)$

type of $\mathcal{A}$	Normal form for prolongable 3-dim Lie algebra $\mathcal{A} = (a, b, c)$ endowed with $a + (b)$	$dim(Aut)$	prolongation
$sl(2)$	$[b, a] = c$ $[b, c] = b$ $[a, c] = -a$	0	$\mathcal{A}_{\infty,1}$
$(3, 2, 0)$ $dim\mathcal{A}^2 = 2$ $\left(\mathcal{A}^2\right)^2 = 0$	$[a, b] = -c \quad [b, c] = 0$ $[a, c] = t_1 b + t_2 c$ $t_1, t_2 \in \mathbb{R},\ t_1 \neq 0$	2	$\mathcal{A}_{\infty,2}$
$(3, 1, 0)$ non-nilp. $dim\mathcal{A}^2 = 1$ $[\mathcal{A}, \mathcal{A}^2] \neq 0$	$\begin{array}{lcl} [b, a] = c & & [b, a] = c \\ [b, c] = 0 & \text{and} & [b, c] = c \\ [a, c] = tc & & [a, c] = 0 \end{array}$	2	$\mathcal{A}_{\infty,2}$
$(3, 1, 0)$ nilp. (Heisenberg) $dim\mathcal{A}^2 = 1$ $[\mathcal{A}, \mathcal{A}^2] = 0$	$[b, a] = c$ $[b, c] = 0$ $[a, c] = 0$	2	$\mathcal{A}_{\infty,2}$

Homogeneous bracket generating affine distributions $A+(B)$ on $\mathbb{R}^4$ with 5-dimensional symmetry algebra

Theorem. If the symmetry algebra of a homogeneous bracket generating affine distribution germ $D = A + (B)$ on $\mathbb{R}^4$ is 5-dimensional then, as an endowed Lie algebra it is isomorphic to the Lie algebra

$$\mathcal{A}_{5,\theta}: \quad \begin{array}{llll} [b,a]=c & [a,c]=\theta b & [w,b]=b & [w,a]=0 \\ [b,c]=d & [a,d]=0 & [w,c]=c & [w,d]=2d \\ [b,d]=0 & [c,d]=0 & & \end{array}$$

endowed with a line $a+(b)$ and isotropy subalgebra (w). The parameter $\theta \in \mathbb{R}$ is an invariant of the endowed algebra.

Corollary. The distribution D can be described by the 4-dimensional subalgebra (a,b,c,d) of $\mathcal{A}_\theta$ endowed with the line $a+(b)$. The parameter θ is an invariant of D with respect to diffeomorphisms.

Corollary. The distribution D is diffeomorphic to

$$B = \frac{\partial}{\partial x_1}, \quad A = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + \frac{1}{2}\left(x_1^2 - \theta x_3^2\right)\frac{\partial}{\partial x_4}$$

4-dim Lie algebras $\mathcal{A}$ admitting a generating line $a+(b)$ and distinguishing prolongable endowed algebras

<table>
<tr><th>4-dim Lie algebra
$\mathcal{A}$ admitting a
generating $a+(b)$</th><th># moduli
in classif.
of $\mathcal{A}$
and classif.
of endowed $\mathcal{A}$</th><th>endowed algebra
is prolongable
if and only if</th><th>Prolongation
(if prolongable)</th></tr>
<tr><td>$so(3)\oplus\mathbb{R}$</td><td>0; 2</td><td>never prolongable</td><td></td></tr>
<tr><td>$sl(2)\oplus\mathbb{R}$</td><td>0; 2</td><td>$dim(b,[b,\mathcal{A}])=2$</td><td>$\mathcal{A}_{\infty,1}$</td></tr>
<tr><td>$(4,3,1,0)$
$dim\mathcal{A}^2=3$
$dim\left(\mathcal{A}^2\right)^2=1$</td><td>1; 2</td><td>$b\in\mathcal{A}^2$</td><td>$\mathcal{A}_{5,\theta}$</td></tr>
<tr><td>$(4,3,0)$
$dim\mathcal{A}^2=3$
$\left(\mathcal{A}^2\right)^2=0$</td><td>2; 2</td><td>$b\in\mathcal{A}^2$</td><td>$\mathcal{A}_{\infty,1}$</td></tr>
<tr><td rowspan="4">$(4,2,0)$
$dim\mathcal{A}^2=2$
$\left(\mathcal{A}^2\right)^2=0$</td><td rowspan="4">non-degen.:
0; 2

degen.:
1; 1

see (*)</td><td>one of the following:
for the operator $ad\ b|_{\mathcal{A}^2}$
and its eigenvalues λ_1,λ_2:</td><td></td></tr>
<tr><td>$ad\ b|_{\mathcal{A}^2}$ is scalar</td><td>$\mathcal{A}_{\infty,1}$</td></tr>
<tr><td>$\lambda_1=\lambda_2=0$,
$ad\ b|_{\mathcal{A}^2}\neq 0$</td><td>$\mathcal{A}_{5,\theta}$</td></tr>
<tr><td>$\lambda_1:\lambda_2=1:2$</td><td>$\mathcal{A}_{5,\theta}$</td></tr>
</table>

(*) a (4,2,0) Lie algebra $\mathcal{A}$ is degenerate if there exists $x\notin\mathcal{A}^2$ such that $ad\ x|_{\mathcal{A}^2}=0$. In this case the condition in the last column is a condition on $\mathcal{A}$.

Classification of homogeneous affine distributions $A+(B)$ on $\mathbb{R}^4$
$=$ classification of non-prolongable 4-dim Lie algebras
endowed with generating affine line $a+(b)$

4-dim Lie algebra $\mathcal{A}=(a,b,c,d)$	normal form for $\mathcal{A}$ endowed with $a+(b)$	parameters
$so(3)\oplus\mathbb{R}$	$[b,a]=c$ $\quad$ $[a,c]=d$ $[b,c]=-a+t_1d$ $\quad$ $[c,d]=t_2a$ $[b,d]=-t_1t_2c$ $\quad$ $[d,a]=t_2c$	$t_1\geq 0$ $t_2>0$
	$[a,b]=-t_1c$ $\quad$ $[b,c]=d$ $[a,c]=t_1b$ $\quad$ $[c,d]=b$ $[a,d]=0$ $\quad$ $[d,b]=c$	$t_1>0$
$sl(2)\oplus\mathbb{R}$	$[b,a]=c$ $\quad$ $[a,c]=d$ $[b,c]=-a+t_1d$ $\quad$ $[c,d]=t_2a$ $[b,d]=-t_1t_2c$ $\quad$ $[d,a]=t_2c$	$t_1\geq 0$, $t_2<0$
	$[a,b]=-t_1c$ $\quad$ $[b,c]=d$ $[a,c]=t_1b$ $\quad$ $[c,d]=-b$ $[a,d]=0$ $\quad$ $[d,b]=c$	$t_1>0$
	$[b,a]=c$ $\quad$ $[a,d]=c$ $[b,c]=d$ $\quad$ $[a,c]=-d$ $[b,d]=0$ $\quad$ $[d,c]=-a$	
$(4,3,1,0)$ $dim\mathcal{A}^2=3$ $dim\left(\mathcal{A}^2\right)^2=1$	$[b,a]=c$ $\quad$ $[a,c]=d$ $[b,c]=\pm a+\lambda c+t_1d$ $[a,d]=0$ $[b,d]=\lambda d$ $\quad$ $[c,d]=0$	$\lambda\geq 0$, $\pm$: invariants of $\mathcal{A}$ $t_1\in\mathbb{R}$
$(4,3,0)$ $dim\mathcal{A}^2=3$ $\left(\mathcal{A}^2\right)^2=0$	$[b,a]=c$ $\quad$ $[a,c]=0$ $[b,c]=d,$ $\quad$ $[a,d]=0$ $[b,d]=a+\lambda_1c+\lambda_2d$ $[c,d]=0$	$\lambda_1,\lambda_2\in\mathbb{R}$: invariants of $\mathcal{A}$
$(4,2,0)$: $dim\mathcal{A}^2=2$ $\left(\mathcal{A}^2\right)^2=0$	$[b,a]=c$ $\quad$ $[a,c]=t_1c$ $[b,c]=d,$ $\quad$ $[a,d]=t_1d$ $[b,d]=t_2c+d$ $\quad$ $[c,d]=0$	$t_1\in\mathbb{R}$ $t_2\neq-\frac{2}{9}$ see (*)
	$[b,a]=c$ $\quad$ $[a,c]=t_1c$ $[b,c]=d,$ $\quad$ $[a,d]=t_1d$ $[b,d]=\pm c$ $\quad$ $[c,d]=0$	$\pm$: invariant of $\mathcal{A}$

(*) For $(4,2,0)$ Lie algebras: if $t_1\neq 0$ the classification of $\mathcal{A}$ is discrete: there are exactly three cases corresponding to $t_1<-\frac{1}{4},\ t_1=\frac{1}{4},\ t_1>-\frac{1}{4}$; If $t_2=0$ then t_1 is as invariant of $\mathcal{A}$.

Prolongable 4-dim Lie algebras endowed with generating affine line $a+(b)$

Normal form for prolongable 4-dim Lie algebra $\mathcal{A}=(a,b,c,d)$ endowed with $a+(b)$	type of $\mathcal{A}$	$dim(Aut)$	prolongation
$[b,a]=c$ $\quad [a,c]=d$ $[b,c]=b$ $\quad [a,d]=t_1c$ $[b,d]=c$ $\quad [c,d]=t_1b+d$ $t_1\in\mathbb{R}$	$sl(2)\oplus\mathbb{R}$	0	$\mathcal{A}_{\infty,1}$
$[b,a]=c$ $\quad [a,c]=d$ $[b,c]=0$ $\quad [a,d]=t_1b+t_2c+t_3d$ $[b,d]=0$ $\quad [c,d]=0$ $t_1,t_2,t_3\in\mathbb{R}$	$t_1\neq 0$: $(4,3,0)$ $t_1=0$: $(4,2,0)$	2	$\mathcal{A}_{\infty,1}$
$[b,a]=c$ $\quad [a,c]=d$ $[b,c]=c$ $\quad [a,d]=t_1c+t_2d$ $[b,d]=d$ $\quad [c,d]=0$ $t_1,t_2\in\mathbb{R}$	$(4,2,0)$	0	$\mathcal{A}_{\infty,1}$
$[b,a]=c$ $\quad [a,c]=t_1c+t_2b$ $[b,c]=d$ $\quad [a,d]=t_1d$ $[b,d]=0$ $\quad [c,d]=0$ $t_1,t_2\in\mathbb{R}$	$t_1\neq 0$: $(4,3,1,0)$ $t_1=0$: $(4,2,0)$	1	$\mathcal{A}_{5,\theta}$ $\theta=t_2-\frac{1}{4}t_1^2$
$[b,a]=c$ $\quad [a,c]=t_1c$ $[b,c]=d$ $\quad [a,d]=t_1d$ $[b,d]=-\frac{2}{9}c+d$ $\quad [c,d]=0$ $t_1\in\mathbb{R}$	$(4,2,0)$	0	$\mathcal{A}_{5,\theta}$ $\theta=-\frac{1}{4}t_1^2$