Heat kernel small time asymptotics at the cut locus in sub-Riemannian geometry

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Introduction

Sub-Riemannian geometry:

- generalization of Riemannian geometry under non holonomic constraints
- geometry underlying the theory of hypoelliptic operators

Main motivation:

- understand the interplay between

$\longrightarrow$ the geometry of these spaces (optimality of geodesics, curvature)

$\longrightarrow$ the analysis of the diffusion processes on the manifold (heat equation)

Questions:

- Which is the “right” Laplace operator associated with the geometry?
- How to relate geometric properties and the heat kernel?

Outline

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1. Sub-Riemannian geometry
2. Sub-Laplacian and Heat Equation
3. Examples

Sub-Riemannian manifolds

Definition

A *sub-Riemannian manifold* is a triple $S = (M, \mathcal{D}, \langle\cdot,\cdot\rangle)$, where

- (*i*) M is a connected smooth manifold of dimension $n \geq 3$;
- (*ii*) $\mathcal{D}$ is a smooth distribution of (*constant*) rank $k < n$, i.e. a smooth map that associates to $q \in M$ a k-dimensional subspace $\mathcal{D}_q$ of T_qM.
- (*iii*) $\langle\cdot,\cdot\rangle_q$ is a Riemannian metric on $\mathcal{D}_q$, that is smooth as function of q.

- The set of *horizontal* vector fields on M, i.e.

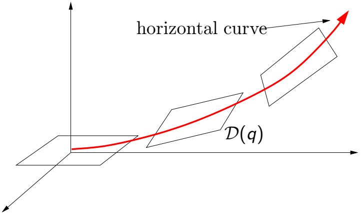
$$\overline{\mathcal{D}} = \{X \in Vec\, M \mid X(q) \in \mathcal{D}_q, \forall\, q \in M\}.$$

satisfies the Lie bracket generating condition $\mathrm{Lie}_q\overline{\mathcal{D}} = T_qM$, $\forall\, q$.

Sub-Riemannian distance

- For a horizontal curve $\gamma : [0, T] \to M$

$$\ell(\gamma) = \int_0^T \sqrt{\langle \dot\gamma(t), \dot\gamma(t) \rangle}\, dt.$$



- The Carnot-Caratheodory distance induced by the sub-Riemannian structure on M is

$$d(q, q') = \inf\{\ell(\gamma) \mid \gamma(0) = q, \gamma(T) = q', \gamma \text{ horizontal}\}.$$

- (M, d) is a metric space and $d(\cdot, \cdot)$ is finite and continuous with respect to the topology of M (Chow's Theorem)

Orthonormal frame

Locally, the pair $(\mathcal{D}, \langle\cdot,\cdot\rangle)$ can be given by assigning a set of k smooth vector fields, called a *local orthonormal frame*, spanning $\mathcal{D}$ and that are orthonormal

$$\mathcal{D}_q = \mathrm{span}\{X_1(q), \dots, X_k(q)\}, \qquad \langle X_i(q), X_j(q)\rangle = \delta_{ij}.$$

The problem of finding geodesics, i.e. curves that minimize the length between two given points q_0, q_1, is equivalent to the optimal control problem

$$\begin{cases} \dot{q} = \displaystyle\sum_{i=1}^{k} u_i X_i(q) & \longleftarrow \text{linear in control} \\ \displaystyle\int_0^T \sum_{i=1}^{k} u_i^2 \to \min & \longleftarrow \text{quadratic cost} \\ q(0) = q_0, \quad q(T) = q_1 \end{cases}$$

Pontryagin Maximum Principle

Theorem (PMP)

If $(u(t), q(t))$ is optimal, there exists a lift $p(t) \in T^*_{q(t)}M$ and $\nu \leq 0$ such that $(q(t), p(t))$ is the solution of the Hamiltonian system associated with

$$\mathcal{H}^\nu(p,q,u) = \sum_{i=1}^k \langle p, X_i(q)\rangle - \frac{\nu}{2}\sum_{i=1}^k u_i^2$$

$$\mathcal{H}^\nu(p(t),q(t),u(t)) = \max_w \mathcal{H}^\nu(p(t),q(t),w) \tag{1}$$

- $\nu = 1$ (extr. normal): the condition (1) implies that in fact $(q(t), p(t))$ is a solution of $H(p,q) = \frac{1}{2}\sum_{i=1}^k \langle p, X_i(q)\rangle^2$
- normal extremals are smooth
- we parametrize extremals starting from q_0 with

$$\Lambda_{q_0} = \{p_0 \in T^*_{q_0}M, H(p_0,q_0) = \frac{1}{2}\} \simeq S^{k-1} \times \mathbb{R}^{n-k}$$

- they are local minimizers

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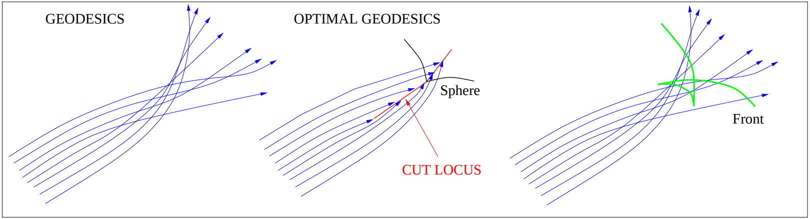
- $\nu = 0$ (extr. abnormal): are abnormal minimizers smooth? open problem $\longrightarrow$ see Monti's talk

Remark: In general a trajectory can be normal and abnormal.

In what follows, we assume no abnormal minimizers that are not normal!

Basic features in SRG - 1

The cut and the conjugate locus starting from a point can be defined analogously to the Riemannian case

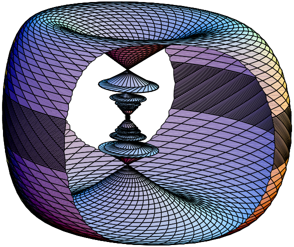


- Conjugate Locus: set of point where geodesics lose local optimality
- Cut locus: set of points where geodesics lose global optimality (and the distance is not smooth)
- Ball of radius R: points reached optimally in time $T \leq R$

Basic features in SRG - 2

Consider geodesics from a point $x_0 \in M$

- there are geodesics loosing optimality arbitrarily close to x_0
- $x \mapsto d^2(x_0, x)$ is not smooth in x_0



Theorem (Agrachev, after Trelat-Rifford)

Let $x_0 \in M$ and $E(x) = \frac{1}{2} d^2(x_0, x)$. Then

- *E is smooth on the open dense set $\Sigma \subset M$*

$$\Sigma(x_0) = \{x \in M \,|\, \exists! \text{ strictly normal non-conjugate minimizer from } x_0 \text{ to } x\}$$

Hausdorff dimension

Define $\mathcal{D}^1 := \mathcal{D}, \qquad \mathcal{D}^2 := \mathcal{D} + [\mathcal{D}, \mathcal{D}], \qquad \mathcal{D}^{i+1} := \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}].$

- Hörmander cond. $\rightarrow \exists\, m \in \mathbb{N}$ (*step*) s.t. $\mathcal{D}^m_q = T_q M$
- *growth vector* of the structure is the sequence

$$\mathcal{G}(S) := (\underset{\substack{\shortparallel \\ k}}{\dim \mathcal{D}}, \dim \mathcal{D}^2, \ldots, \underset{\substack{\shortparallel \\ n}}{\dim \mathcal{D}^m})$$

- Mitchell Theorem: the Hausdorff dimension of (M, d) is given by the formula

$$Q = \sum_{i=1}^{m} i k_i = k_1 + 2k_2 + \ldots + m k_m,$$

The Hausdorff dimension is bigger than the topological one: $Q > n$

Sub-Laplacian

The sub-Laplacian operator Δ on $(M, \langle\cdot,\cdot\rangle, \mathcal{D})$ is the natural generalization of the Laplace-Beltrami operator and is defined as follows

$$\Delta\phi = div(grad\phi)$$

- $grad\phi$ is the unique *horizontal* vector field dual to $d\phi\big|_{\mathcal{D}}$ (use the *metric*)

$$grad\phi = \sum_{i=1}^{k} X_i(\phi)X_i$$

- The divergence of a vector field X says how much the flow of X change a *volume* μ, i.e. $L_X\mu = (div\, X)\mu$
- The Leibnitz rule for *div* gives

$$\Delta = \sum_{i=1}^{k} X_i^2 + (div\, X_i)X_i$$

$\longrightarrow$ sum of squares + first order terms that depend on the volume

We assume that a smooth volume μ is fixed!

Which volume?

The choice of a canonical volume is related with the definition of a canonical sub-Laplacian.

- in the Riemannian case the metric canonically defines the Riemannian volume
- being a metric space, one can consider the (spherical) Hausdorff volume
- in this case they are proportional

In the sub-Riemannian case the metric does not define a canonical volume in principle

- in the equiregular case we can use the structure of the Lie bracket to define a volume, called Popp volume
- the spherical Hausdorff volume is not proportional and in general, it is not even smooth

$\rightarrow$ See Gauthier's talk.

Heat equation

Sub-Riemannian heat equation on a *complete* manifold M with smooth measure μ

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(t,x) = \Delta\psi(t,x), & \text{in } (0,\infty)\times M, \\ \psi(0,x) = \varphi(x), & x \in M, \quad \varphi \in C_0^\infty(M). \end{cases} \qquad (*)$$

where $\psi(0,x) = \lim_{t\to 0} \psi(t,x)$. Recall that

$$\Delta = \sum_{i=1}^{k} X_i^2 + (div\, X_i)X_i$$

Theorem (Hörmander)

If $X_1,\dots,X_k$ are bracket generating then $\Delta = \sum_{i=1}^k X_i^2 + X_0$ is hypoelliptic.

Heat kernel

For $f, g \in C_0^\infty(M)$ with compact support, the standard divergence theorem

$$\int_M f\,(div\,X)\,d\mu = -\int_M Xf\,d\mu, \qquad \text{(no metric required!)}$$

applied to $X = grad\,g$ gives

$$\int_M f\,\Delta g\,d\mu = -\int_M \langle grad\,f, grad\,g\rangle\,d\mu$$

implies that Δ is symmetric and negative $\rightarrow$ Δ essentially self-adjoint on $C_0^\infty(M)$.

The problem $(*)$ has a unique solution for every initial datum $\varphi \in C_0^\infty(M)$

$$\psi(t,x) := e^{t\Delta}\varphi(x) = \int_M p_t(x,y)\varphi(y)d\mu(y), \qquad \varphi \in C_0^\infty(M),$$

where $p_t(x,y) \in C^\infty$ is the so-called *heat kernel* associated to Δ.

Results on the asymptotic of $p_t(x, y)$

Fix $x, y \in M$, $\dim M = n$:

Theorem (Main term, Leandre)

$$\lim_{t \to 0} 4t \log p_t(x, y) = -d^2(x, y) \tag{1}$$

Theorem (Smooth points, Ben Arous)

Assume $y \in \Sigma(x)$, then

$$p_t(x, y) \sim \frac{1}{t^{n/2}} \exp\left(-\frac{d^2(x, y)}{4t}\right) \tag{2}$$

Facts

1. In Riemannian geometry $x \in \Sigma(x)$, in sub-Riemannian is not true!
2. The on-the-diagonal expansion indeed is different.

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Theorem (On the diagonal, Ben Arous)

We have the expansion

$$p_t(x, x) \sim \frac{1}{t^{Q/2}} \tag{2}$$

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Results on the asymptotic of $p_t(x, y)$

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$$p_t(x, y) \sim \frac{1}{t^{n/2}} \exp\left(-\frac{d^2(x, y)}{4t}\right) \tag{2}$$

Questions

1. What happens in (2) if $y \in Cut(x)$?
2. Can we relate the expansion of $p_t(x, y)$ with the properties of the geodesics joining x to y?

Example: Heisenberg

In the Heisenberg group the Heat kernel is explicit (here $q = (x, y, z)$)

$$p_t(0,q) = \frac{1}{(4\pi t)^2} \int_{-\infty}^{\infty} \frac{\tau}{\sinh \tau} \exp\left(-\frac{x^2+y^2}{4t} \frac{\tau}{\tanh \tau}\right) \cos\left(\frac{z\tau}{t}\right) d\tau.$$

and gives the asymptotics for cut-conjugate points $\zeta = (0, 0, z)$

$$p_t(0,\zeta) \sim \frac{1}{t^2} \exp\left(-\frac{\pi z}{t}\right) = \frac{1}{t^2} \exp\left(-\frac{d^2(0,\zeta)}{4t}\right)$$

Remark: The factor $\frac{1}{2}$ that is added confirm the fact that the points $\zeta = (0, 0, z)$ are not smooth points. What is the meaning of the $\frac{1}{2}$?

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$$p_t(0,\zeta)\sim\frac{1}{t^{\frac{3}{2}+\frac{1}{2}}}\exp\left(-\frac{\pi z}{t}\right)=\frac{1}{t^{\frac{3}{2}+\frac{1}{2}}}\exp\left(-\frac{d^2(0,\zeta)}{4t}\right)$$

Remark: The factor $\frac{1}{2}$ that is added confirm the fact that the points $\zeta=(0,0,z)$ are not smooth points. What is the meaning of the $\frac{1}{2}$?

What happens at non good point?

Let $x, y \in M$ with $y \in Cut(x)$ and write

$$p_t(x,y) = \int_M p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z)$$

Idea: assume $z \in \Sigma(x) \cap \Sigma(y)$ and apply Ben-Arous expansion

$$p_{t/2}(x,z)p_{t/2}(z,y) \sim \frac{1}{t^n} \exp\left(-\frac{d^2(x,z)+d^2(z,y)}{4t}\right)$$

This led to the study of an integral of the kind

$$p_t(x,y) = \frac{1}{t^n} \int_M c_{x,y}(z) \exp\left(-\frac{h_{x,y}(z)}{2t}\right) d\mu(z)$$

where $h_{x,y}$ is the hinged energy function

$$h_{x,y}(z) = \frac{1}{2}\left(d^2(x,z)+d^2(z,y)\right).$$

Properties of $h_{x,y}$ hinged energy function

Lemma

Let Γ be the set of midpoints of the minimal geodesics joining x to y.
Then $\min h_{x,y} = h_{x,y}(\Gamma) = d^2(x,y)/4$.

- A minimizer is called strongly normal if any piece of it is strictly normal.

Theorem

Let γ be a strongly normal minimizer joining x and y. Let z_0 be its midpoint. Then

(i) *y is conjugate to x along $\gamma \Leftrightarrow Hess_{z_0} h_{x,y}$ is degenerate.*

(ii) *The dimension of the space of perturbations for which γ is conjugate is equal to* $\dim\ker Hess_{z_0} h_{x,y}$.

Remark: *Hess* $h_{x,y}$ is never degenerate along the direction of the geodesic!

Laplace integrals in $\mathbb{R}^n$

Let $K \subset \mathbb{R}^n$ be a compact set and assume g has a minimum in $\operatorname{int} K$

- The asymptotic of the Laplace integral

$$\int_K f(z)e^{-g(z)/t}dx \quad \text{as } t \searrow 0$$

  is determined by the behavior of the function g near its minimum.

- Example in $\mathbb{R}$ with $g(z) = z^{2m}$

$$I(t) = \int_a^b f(z)e^{-z^{\mathbf{2m}}/t}dz \sim C_m\, f(0)\, t^{1/2m}, \quad t \to 0$$

- Example in $\mathbb{R}^n$ with $g(z) = \sum z_i^{2m_i}$

$$I(t) = \int_a^b f(z)e^{-g(z)/t}dz \sim C\, f(0)\, t^{\sum_i 1/2m_i}, \quad t \to 0$$

Hinged *vs* Asymptotics

Theorem

For a sufficiently small neighborhood $N(\Gamma)$ of the set of midpoints from x to y

$$p_t(x,y) = \frac{1}{t^n}\int_{N(\Gamma)} \exp\left(-\frac{h_{x,y}(z)}{2t}\right)(c_{x,y}(z) + O(t))d\mu(z)$$

Assume that, in a neighborhood of the midpoints of the strongly normal geodesics joining x to y there exists coordinates such that

$$h_{x,y}(z) = \frac{1}{4}d^2(x,y) + z_1^{2m_1} + \ldots + z_n^{2m_n} + o(|z_1|^{2m_1} + \ldots + |z_n|^{2m_n})$$

Then for some constant $C > 0$

$$p_t(x,y) = \frac{1}{t^{n-\sum_i \frac{1}{2m_i}}} \exp\left(-\frac{d^2(x,y)}{4t}\right)(C + o(1)).$$

Note: $h_{x,y}$ non degenerate $(m_i = 2) \rightarrow$ the exponent is $n/2$

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Note: $h_{x,y}$ non degenerate ($m_i = 2$) $\longrightarrow$ the exponent is $n/2$

Example: Heisenberg

In the Heisenberg group we had the asymptotics for cut-conjugate points $\zeta = (0,0,z)$

$$p_t(0,\zeta) \sim \frac{1}{t^2} \exp\left(-\frac{\pi z}{t}\right) = \frac{1}{t^2} \exp\left(-\frac{d^2(0,\zeta)}{4t}\right)$$

Remark: The factor $\frac{1}{2}$ is a consequence of the fact that there exists a one parametric family of optimal trajectories (varying the angle), hence the hinged energy function is actually a function of two variables, being constant on the midpoints.

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Cut/Conjugacy *vs* Asymptotics

Using the geometric properties of $h_{x,y}$ we can state also the following

Corollary

Let M be an n-dimensional complete SR manifold, μ smooth volume. Let $x \neq y$ and assume that every optimal geodesic joining x to y is strongly normal.

- *Then there exist positive constants C_i, such that for small t*

$$\frac{C_1}{t^{n/2}} e^{-d^2(x,y)/4t} \leq p_t(x,y) \leq \frac{C_2}{t^{n-(1/2)}} e^{-d^2(x,y)/4t},$$

- *If x and y are conjugate along at least one minimal geodesic*

$$\frac{C_3}{t^{(n/2)+(1/4)}} e^{-d^2(x,y)/4t} \leq p_t(x,y) \qquad ,$$

- *If x and y are not conjugate*

$$p_t(x,y) = \frac{C_4 + O(t)}{t^{n/2}} e^{-d^2(x,y)/4t},$$

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Remarks

- The precise correspondence between *Hess* $h_{x,y}$ and cut/conjugacy works at order 1.
- To have the precise asymptotic one need that the expansion of $h_{x,y}$ is diagonal in some coordinates.

Nevertheless there are at least two cases that simplifies the analysis

- If we have a one parametric family of optimal trajectories then $h_{x,y}$ is constant along the trajectory of midpoints.
- If there is only one degenerate direction then $h_{x,y}$ is always diagonalizable

Lemma (Splitting Lemma - Gromoll, Meyer)

Let $g : \mathbb{R}^n \to \mathbb{R}$ smooth such that $g(0) = dg(0) = 0$ and that $\dim \ker d^2 g(0) = 1$ with 0 isolated minimum of g.
Then there exists a diffeomorphism ϕ of $\mathbb{R}^n$ and a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$g(\phi(u)) = \sum_{i=1}^{n-1} u_i^2 + \psi(u_n), \qquad \textit{where} \qquad \psi(u_n) = O(u_n^4).$$

Outline

Bi-Heisenberg group

Is the sub-Riemannian structure on $\mathbb{R}^5$ defined by the o.n. frame

$$X_1 = \partial_{x_1} - \frac{\alpha_1}{2} y_1 \partial_z, \qquad X_2 = \partial_{y_1} + \frac{\alpha_1}{2} x_1 \partial_z$$

$$X_3 = \partial_{x_2} - \frac{\alpha_2}{2} y_2 \partial_z, \qquad X_4 = \partial_{y_2} + \frac{\alpha_2}{2} x_2 \partial_z,$$

Remark: If $Z = \partial_z$ then $[X_1, X_2] = \alpha_1 Z$ and $[X_3, X_4] = \alpha_2 Z$,

- we restrict to the contact case $\alpha_1, \alpha_2 > 0$.
- one can always assume $0 < \alpha_1 \leq \alpha_2$
- the case $\alpha_1 < \alpha_2$ and $\alpha_1 = \alpha_2$ are really different!!!

Exponential map

The exponential map starting from the origin $\mathrm{Exp}_0 : \Lambda_0 \times \mathbb{R}^+ \to M$ where

$$\Lambda_0 = \{p_0 = (r_1, r_2, \theta_1, \theta_2, w) \in T_0^* M \mid \; r_1^2 + r_2^2 = 1, \; \theta_1, \theta_2 \in S^1, \; w \in \mathbb{R}\}.$$

The arclength geodesics associated with $p_0 = (r_1, r_2, \theta_1, \theta_2, w) \in \Lambda_0$ and $|w| \neq 0$

$$\begin{aligned} x_i(t) &= \frac{r_i}{\alpha_i w}(\cos(\alpha_i wt + \theta_i) - \cos\theta_i), \\ y_i(t) &= \frac{r_i}{\alpha_i w}(\sin(\alpha_i wt + \theta_i) - \sin\theta_i), \qquad i = 1, 2, \\ z(t) &= \frac{1}{2w^2}(wt - \sum_i \frac{r_i^2}{\alpha_i} \sin\alpha_i wt). \end{aligned} \tag{3}$$

Lemma

Each of these geodesic is optimal up to its cut time

$$t_{cut} = t_{conj} = \frac{2\pi}{|w| \max\{\alpha_1, \alpha_2\}}. \tag{4}$$

Bi-Heisenberg group

Cut locus

- $\alpha < 1 \rightarrow$ the cut locus is a three dimensional set containing the z-axis.
  - Each point is reached by a one parametric family of optimal geodesics
- $\alpha = 1 \rightarrow$ the cut locus degenerates to the z-axis only.
  - Each point is reached by a three parametric family of optimal geodesics

Heat Kernel at a point $\zeta = (0, 0, 0, 0, z)$ on the z-axis

- $\alpha < 1 \rightarrow$ the expansion

$$p_t(0, \zeta) \sim \frac{1}{t^3} \exp\left(-\frac{d^2(0, \zeta)}{4t}\right)$$

- $\alpha = 1 \rightarrow$ the expansion

$$p_t(0, \zeta) \sim \frac{1}{t^4} \exp\left(-\frac{d^2(0, \zeta)}{4t}\right)$$

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Heat Kernel at a point $\zeta = (0, 0, 0, 0, z)$ on the z-axis

- $\alpha < 1 \rightarrow$ the expansion

$$p_t(0, \zeta) \sim \frac{1}{t^{\frac{5}{2}+\frac{1}{2}}} \exp\left(-\frac{d^2(0, \zeta)}{4t}\right)$$

- $\alpha = 1 \rightarrow$ the expansion

$$p_t(0, \zeta) \sim \frac{1}{t^{\frac{5}{2}+\frac{3}{2}}} \exp\left(-\frac{d^2(0, \zeta)}{4t}\right)$$

Application: Grushin plane

- It is the generalized sub-Riemannian structure on $\mathbb{R}^2$ for which an orthonormal frame of vector fields is given by

$$X = \partial_x, \qquad Y = x\partial_y. \tag{5}$$

- we want to compute the expansion of the heat kernel in the Grushin plane at a cut-conjugate point, starting from a Riemannian point.

$$\Delta = X^2 + Y^2 = \partial_x^2 + x^2\partial_y^2$$

Remarks:

- the exponential map from the origin has no cut-conjugate points
- from a Riem point the exponential map is integrable in trigonometric functions and provides a generic structure of cut and conjugate locus.
- the canonical (Riemannian) volume is not smooth (actually the heat does not cross the vertical line in this case [Boscain, Laurent])

- The sub-Riemannian Hamiltonian in $T^*\mathbb{R}^2$ is the smooth function

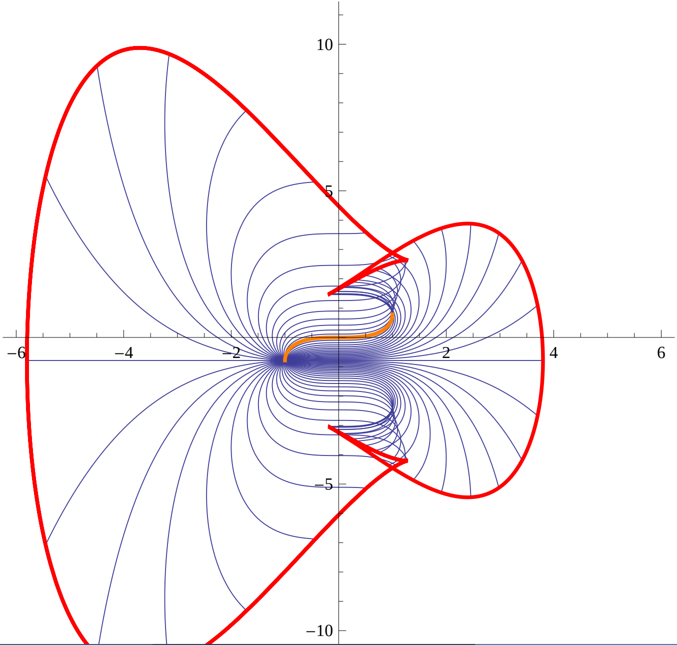
$$H : T^*\mathbb{R}^2 \to \mathbb{R}, \qquad H(p_x, p_y, x, y) = \frac{1}{2}(p_x^2 + x^2 p_y^2). \tag{6}$$

- There are no abormal minimizers
- The arclength geodesic flow $\mathrm{Exp} : \mathbb{R}_+ \times S^1 \to \mathbb{R}^2$
  - $\rightarrow$ starting from the Riemannian point $q_0 = (-1, -\pi/4)$
  - $\rightarrow$ initial condition $(p_x(0), p_y(0)) = (\cos\theta, \sin\theta)$

$$\mathrm{Exp}(t,\theta) = (x(t,\theta), y(t,\theta)),$$

$$\begin{aligned} x(t,\theta) &= -\frac{\sin(\theta - t\sin\theta)}{\sin\theta}, \\ y(t,\theta) &= -\frac{\pi}{4} + \frac{1}{4\sin\theta}\left(2t - 2\cos\theta + \frac{\sin(2\theta - 2t\sin\theta)}{\sin\theta}\right), \end{aligned} \tag{7}$$

- the point $q_1 = (1, \pi/4)$ is a cut-conjugate point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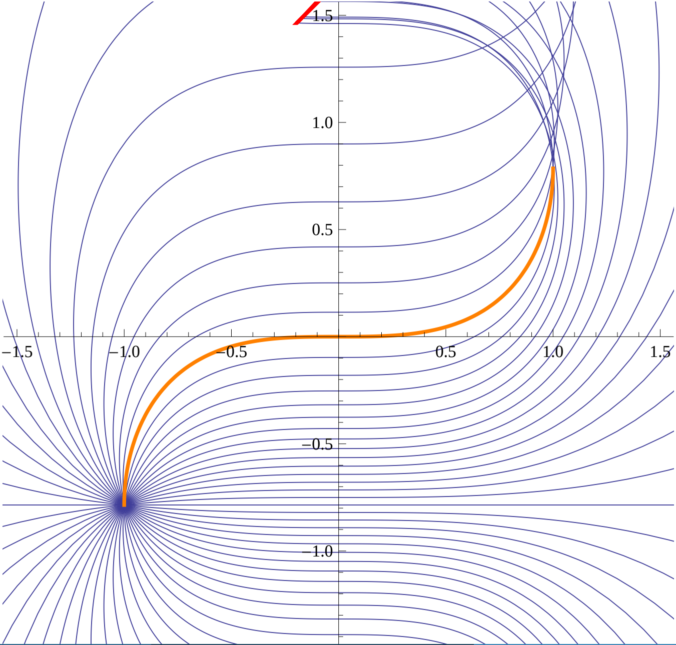
1.5
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−0.5
0.5
1.0
1.5

The hinged energy function

$$\begin{aligned} h(x,y) &= \frac{1}{2}\left(d^2(q_0,(x,y)) + d^2(q_1,(x,y))\right) \\ &= \frac{1}{2}\left(d^2(q_0,(x,y)) + d^2(q_0,(-x,-y))\right), \end{aligned}$$

We found an (explicit!) change of coordinates $(x,y) = \phi(u,v)$ near the origin such that

$$h(\phi(u,v)) = u^2 + v^4 + O(\|(u,v)\|^5)$$

Using that $d(q_0,q_1) = \pi$, this proves

Theorem

The heat kernel $p_t(q_0,q_1)$ satisfies the following asymptotic expansion

$$p_t(q_0,q_1) = \frac{1}{t^{5/4}} exp\left(-\frac{\pi^2}{4t}\right)(C + O(t)) \tag{8}$$

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$$h(x,y) = \frac{1}{2}\left(d^2(q_0,(x,y)) + d^2(q_1,(x,y))\right)$$
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Theorem

The heat kernel $p_t(q_0,q_1)$ satisfies the following asymptotic expansion

$$p_t(q_0,q_1) = \frac{1}{t^{2-\frac{1}{2}-\frac{1}{4}}} exp\left(-\frac{\pi^2}{4t}\right)(C + O(t)) \tag{8}$$

Open problems and related works

→ Expansion at the cut/conjugate locus

- Case of a generic 2D Riemannian and 3D sub-Riemannian
- in the case of a 3D sub-Riemannian the cut locus is adjacent to the origin

→ Heat kernel and geometry

- we obtained an expansion on the diagonal in the 3D

$$p(t,x,x) \sim \frac{1}{16t^2}(1+\kappa(x)t+O(t^2)), \qquad \text{for} \quad t \to 0.$$

- higher dim? Curvature of general sub-Riemannian spaces
  → see Agrachev's talk

→ References:

- D.B., U. Boscain, R. Neel, *Small time asymptotics of the SR heat kernel at the cut locus*, accepted on JDG, on ArXiv
- D.B., U. Boscain, R. *The Bi-Heisenberg group in SR geometry*, in preparation.