Maxwell's equations in Carnot groups

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Researches in collaboration with

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- for a while, for sake of simplicity, we have set all "physical" constants to be 1.

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- Then

in special relativity, Maxwell's equations of electromagnetism in the 4-dimensional free space-time $\mathbb{R}^3_x \times \mathbb{R}_s$ take the form

$$dF = 0 \quad \text{and} \quad d(*_M F) = \mathcal{J}, \tag{1}$$

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- $*_M$ denotes the Hodge duality with respect to usual Minkowskian scalar product;
- $\mathcal{J} = *J \wedge ds - \rho$ is a closed 3-form (source form) in $\mathbb{R}^3_x \times \mathbb{R}_s$, where $\rho(\cdot, s) = \rho_0(\cdot, s)\, dV$ is a volume form on $\mathbb{R}^3$ for any fixed $s \in \mathbb{R}$ and J is a 1-form (the current density).

Introducing a *vector potential* A related F, that is

$$F = dA \quad \text{with} \quad A = A_\Sigma + \phi\, ds$$

where $A_\Sigma = A_{\Sigma,1}(x,s)\,dx_1 + A_{\Sigma,2}(x,s)\,dx_2 + A_{\Sigma,3}(x,s)\,dx_3$ is a 1-form in $\mathbb{R}^3$ and $\phi = \phi(x,s)$ is a scalar function, then Maxwell's equations are equivalent to the following system for the vector potential ($\Box = \partial_s^2 - \Delta_x$):

$$\begin{aligned} \Box A_\Sigma &= J \\ \Box \phi &= \rho \end{aligned}$$

provided A satisfies the Lorenz gauge condition

$$*_M d *_M A = 0.$$

We want to develop a similar geometric theory in Carnot groups leading (hopefully) to a “natural” (intrinsic) wave equation in Carnot groups.

To fix our notations:

A *Carnot group* $\mathbb{G}$ *of step* κ is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ of the left invariant vector fields admits a *step* κ *stratification*, i.e. there exist linear subspaces $V_1, ..., V_\kappa$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, W]$ with $X \in V_1$ and $W \in V_i$.

From now on, we fix a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ adapted to the stratification, as well as a scalar product $\langle \cdot, \cdot \rangle$ making the basis orthonormal.
Adapted means that $\{X_1, \dots, X_{m_1}\}$ is a basis of V_1, $\{X_{m_1+1}, \dots, X_{m_2}\}$ is a basis of V_2, and so on.

The scalar product defines naturally a scalar product on the spaces $\Lambda_k\mathfrak{g}$ of k-vectors, and on the spaces $\Lambda^k\mathfrak{g}$ of k-covectors.

M. Rumin (CRAS 1999) defines a suitable sub–complex (E_0^*, d_c) of de Rham complex (Ω^*, d), in which essentially the forms are sections of a suitable fiber bundle generated by left translations:

$$\{0\} \to C^\infty(\mathbb{G}) \equiv E_0^0(\mathbb{G}) \xrightarrow{d_c} E_0^1(\mathbb{G}) \xrightarrow{d_c} \cdots \xrightarrow{d_c} E_0^n(\mathbb{G}) \to \{0\},$$

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- $E_0^1 = \{\text{horizontal 1–forms}\}$;
- $d_c f$ is the horizontal differential of f for $f \in E_0^0(\mathbb{G})$;
- (E_0, d_c) is Hodge-self-dual, in the sense that $*E_0^k = E_0^{n-k}$.

In general, the differential d_c is defined through a suitable subcomplex (E^*, d) of the De Rham complex (Ω^*, d) (we refer to (E^*, d) as to the complex of "lifted forms"). Il Π_E denotes a suitable projection on E^* and Π_{E_0} is the orthogonal projection on E_0^*, then

$$d_c = \Pi_{E_0} d \Pi_E.$$

- The following diagram gives a synopsis of the construction

$$
\begin{tikzcd}
\cdots \arrow[r, "d_c"] & E_0^h \arrow[r, "d_c"] \arrow[d, "\Pi_E"'] & E_0^{h+1} \arrow[r, "d_c"] & \cdots \\
\cdots \arrow[r, "d"] & E^h \arrow[r, "d"] \arrow[d, "i"'] & E^{h+1} \arrow[r, "d"] \arrow[u, "\Pi_{E_0}"'] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "d"] & \Omega^h \arrow[r, "d"] & \Omega^{h+1} \arrow[r, "d"] & \cdots
\end{tikzcd}
$$

Roughly speaking, the projection Π_E is meant to remove the obstructions to the exactness of the complex, whereas the projection Π_{E_0} to minimize the number of the closeness conditions

A crucial property of the complex (E_0^*, d_c) is that in general, d_c is an operator of higher order in the horizontal derivatives. In addition, it may fail to be homogeneous.

Theorem (F., Tesi, 2010)

Let $\mathbb{G}$ be a free group of step κ. Then all forms in E_0^1 have weight 1 and all forms in E_0^2 have weight $\kappa + 1$.

- Obviously, the naif Laplacian associated with d_c, i.e.

$$\delta_c d_c + d_c \delta_c,$$

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- Even if d_c is homogeneous, as in $\mathbb{H}^n$ case, the associated natural Laplacian could be not homogeneous. For instance, on 1–forms in $\mathbb{H}^1$, $\delta_c d_c$ is a 4th order operator, while $d_c \delta_c$ is a 2nd order one. This is due to the fact that the order of d_c depends on the order of the forms on which it acts on.

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- Indeed, d_c on 1–forms in $\mathbb{H}^1$ is a 2nd order operator, as well as its adjoint δ_c (which acts on 2–form), while δ_c on 1–forms is a first order operator, since it is the adjoint of d_c on 0–forms, which is a first order operator.

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- Things become more complicated in general Carnot groups.

In $\mathbb{H}^1$, Rumin proved that the naif Laplacian $\delta_c d_c + d_c \delta_c$ on intrinsic 1-forms can be profitably replaced by the 4th-order differential operator in the horizontal derivatives

$$\Delta_{\mathbb{H},1} := \delta_c d_c + (d_c \delta_c)^2,$$

while on 0-forms $\delta_c d_c + d_c \delta_c := \Delta_{\mathbb{H},0} = -\Delta_{\mathbb{H}} := -(X^2 + Y^2)$.

Theorem (Rumin, 1994)

The operator $\Delta_{\mathbb{H},1}$ *is maximal hypoelliptic, i.e, if* $\Omega \subset \mathbb{H}^1$ *is a bounded open set, then the* $L^2(\Omega)$*-norm of all 4th order horizontal derivatives of an intrinsic 1-form* α *supported in* Ω *are controlled by*

$$\|\Delta_{\mathbb{H},1}\alpha\|_{L^2(\Omega)} + \|\alpha\|_{L^2(\Omega)}.$$

Notice that, unlike the Euclidean Laplacian on forms, the operator $\Delta_{\mathbb{H},1}$ fails to be diagonal.

Analogously, if $\mathbb{G}$ is a free group of step κ we set

$$\Delta_{\mathbb{G},1} := \delta_c d_c + (d_c \delta_c)^\kappa,$$

and we have

Theorem (F., Tesi, 2009)

Let $\mathbb{G}$ be a free group of step κ. The operator $\Delta_{\mathbb{G},1}$ is maximal hypoelliptic, i.e, if $\Omega \subset \mathbb{G}$ is a bounded open set, then the $L^2(\Omega)$-norm of all horizontal derivatives of order 2κ of an intrinsic 1-form α supported in Ω are controlled by

$$\|\Delta_{\mathbb{G},1}\alpha\|_{L^2(\Omega)} + \|\alpha\|_{L^2(\Omega)}.$$

Back to Maxwell's equations:

Notice if $\mathbb{G}$ is a Carnot group, then $\mathbb{R} \times \mathbb{G}$ is a Carnot group.

The Lie algebra $\tilde{\mathfrak{g}}$ of of $\mathbb{R} \times \mathbb{G}$ admits the stratification

$$\tilde{\mathfrak{g}} = \tilde{V}_1 \oplus V_2 \oplus \cdots \oplus V_\kappa,$$

where $\tilde{V}_1 = \operatorname{span}\{S, V_1\}$, where $S = \frac{\partial}{\partial s}$.

Since the adapted basis $\{X_1, \dots, X_n\}$ has been already fixed once and for all, the associated fixed basis for $\mathbb{R} \times \mathbb{G}$ will be $\{S, X_1, \dots, X_n\}$.

If $\mathbb{G}$ is a Carnot group, we can formulate Maxwell's equations in $\mathbb{R} \times \mathbb{G}$ as follows

$$d_c F = 0 \quad \text{and} \quad d_c(*_M F) = \mathcal{J}, \tag{2}$$

where

- $F = E \wedge ds + B \in E^2_{0,\mathbb{R}\times\mathbb{G}}$ is an intrinsic 2-form in $\mathbb{R} \times \mathbb{G}$;

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- $\mathcal{J} = *J \wedge ds - \rho$ is a closed 3-form in $\mathbb{R}\times\mathbb{G}$, , where $\rho(\cdot, s) = \rho_0(\cdot, s)\, dV$ is a volume form on $\mathbb{G}$ for any fixed $s \in \mathbb{R}$ and $J \in E^1_{0,\mathbb{R}\times\mathbb{G}}$ is an (intrinsic) 1-form.

Maxwell's equations in $\mathbb{G} \times \mathbb{R}$ are invariant under *contact Lorentz transformations*.

Invariance is due to the fact that the pull-back induced by a contact Lorentz matrix L commutes with both d_c and the Hodge operator $*_M$.

A homogeneous homomorphism $L : \mathbb{G} \to \mathbb{G}$ induces a contact linear map $L : \mathfrak{g} \to \mathfrak{g}$, i.e. a map L such that

$$L(V_i) \subset V_i, \quad i = 1, \dots, \kappa.$$

We say that L is a contact Lorentz transformation if

$${}^tLGL = G,$$

where

$$G = \left(\begin{array}{cccc|c|c}
\begin{matrix} -1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 0 & 1 \end{matrix} & & & & 0 & 0 \\
\hline
 & 0 & & & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\
\hline
 & 0 & & & 0 & \ddots
\end{array}\right)$$

Given a group $\mathbb{G}$, we refer to $HO_{\mathbb{G}}$ as to the contact Lorentzian group of $\mathbb{G}$.

Example

Take $\mathbb{G} = \mathbb{H}^1$. A matrix L belongs to $HO(\mathbb{G})$ if and only if it has the form

$$L = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & \det A \end{pmatrix},$$

where

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{is a unitary matrix.}$$

Looking for solutions of Maxwell's equations

$$d_c F = 0 \quad \text{and} \quad d_c(*_M F) = \mathcal{J},$$

since the complex $(E^*_{0,\mathbb{R}\times\mathbb{G}}, d_c)$ is exact, we are lead to look for a vector potential (i.e. an intrinsic 1-form on $\mathbb{G} \times \mathbb{R}$ such that $F = d_c A$) of the form

$$A = A_\Sigma + \phi\, ds,$$

satisfying a suitable gauge condition.

We are therefore interested in formulating and studying a "natural" (intrinsic) wave equation for the vector potential, in analogy with the Euclidean setting.

In the simplest case of the first Heisenberg group we have

Theorem (intrinsic wave equation)

Suppose $F \in E^2_{0,\mathbb{H}^1\times\mathbb{R}}$ *satisfies Maxwell's equations. Then* $F = d_c A$ *with* $A = A_1 dx + A_2 dy + \phi\, ds := A_\Sigma + \phi\, ds \in E^1_{0,\mathbb{H}^1\times\mathbb{R}}$. *In addition, A satisfies*

$$\begin{aligned} \frac{\partial^2 A_\Sigma}{\partial s^2} &= -\Delta_{\mathbb{H},1} A_\Sigma + J && (3) \\ \frac{\partial^2 \phi}{\partial s^2} &= -\Delta^2_{\mathbb{H}}\phi + \frac{1}{16}\Delta_{\mathbb{H}}\rho_0, && (4) \end{aligned}$$

where $\Delta_{\mathbb{H}} := X^2 + Y^2 (= -\Delta_{\mathbb{H},0})$ *is the usual subelliptic Laplacian in* $\mathbb{H}^1$*, provided the following gauge condition holds:*

$$d_c^* d_c d_c^* A_\Sigma + \frac{\partial \phi}{\partial s} = 0. \tag{5}$$

Notice the gauge condition (5) can be always satisfied replacing the potential A by the potential $A + d_c\psi$, with ψ satisfying

$$\frac{\partial^2 \psi}{\partial s^2} = -\Delta^2_{\mathbb{H}} \psi - \left(d_c^* d_c d_c^* A_\Sigma + \frac{\partial \phi}{\partial s} \right). \tag{6}$$

An analogous result holds in free groups $\mathbb{G}$ of step κ. The new "wave equation" becomes

$$\begin{aligned} \frac{\partial^2 A_\Sigma}{\partial s^2} &= -\Delta_{\mathbb{G},1} A_\Sigma + J \\ \frac{\partial^2 \phi}{\partial s^2} &= -(-\Delta_{\mathbb{G}})^\kappa \phi - (-\Delta_{\mathbb{G}})^{\kappa-1} \rho_0, \end{aligned}$$

where $\Delta_{\mathbb{G}} := \sum_{j=1}^m X_j^2$ is the usual subelliptic Laplacian in $\mathbb{G}$.

We stress that, in our wave equation, in the homogeneous Laplacian $\Delta_{\mathbb{G},1}$, the "natural part" $\delta_c d_c$ is generated by Maxwell equation, whereas the "artificial part" $(d_c\delta_c)^\kappa$ is generated by the gauge condition, that is arbitrary chosen according our convenience.

If the source form satisfies suitable conditions at the infinity, then the solutions of our wave equation generate solutions of Maxwell equations

The existence of solutions of equations (3), (4) and (5) can be obtained by means of abstract arguments of "infinitesimal generators of cosine functions".

Theorem

Let $I \subset \mathbb{R}$ be a bounded interval such that $0 \in I$. Let $\alpha_0 \in W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$, $\alpha_1 \in L^2(\mathbb{G}, E_0^1)$ and $J \in L^2(I, L^2(\mathbb{G}, E_0^1))$. Then there exists a unique strong solution

$$\alpha \in C^0(I, W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)) \cap C^1(I, L^2(\mathbb{G}, E_0^1))$$

of the linear equation

$$\begin{cases} \partial_s^2 \alpha + \Delta_{\mathbb{G},1}\alpha = J, & \text{for } s > 0 \\ \alpha_{|_{s=0}} = \alpha_0, \quad \alpha_{s|_{s=0}} = \alpha_1 \end{cases} \tag{7}$$

If in addition $J \in C^0(I, W^{-r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1))$, then $\alpha \in C^2(I, W^{-r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1))$.

The propagators are explicitly given by

$$C(s) = \int_{-\infty}^{0} \cos(s|\lambda|^{1/2})\, dE(\lambda)$$

and

$$S(s) = \int_{-\infty}^{0} |\lambda|^{-1/2} \sin(s|\lambda|^{1/2})\, dE(\lambda),$$

where $dE(\lambda)$ is the spectral measure associated with $-\Delta_{\mathbb{G},1}$.

Again in $\mathbb{H}^1$, it is interesting to notice that, if we look for solutions $u = u(t; x, y)$ that do not depend on the variables of higher layer, we fall in the equation of the classical elasticity

$$\frac{\partial^2 u}{\partial s^2} = -\Delta^2 u, \tag{8}$$

the so-called Germain-Lagrange equation for the vibration of plates.

We want to show that our group equations can be seen as “limit equations” for strongly anisotropic media.
For sake of simplicity, let us restrict ourselves to $\mathbb{H}^1$.

Let us go back to classical Maxwell's equations in $\mathbb{R} \times \mathbb{R}^3$ in the matter".

The physical properties of the matter are described by two 3×3 matrices $[\varepsilon], [\mu]$ that are called dielectric permittivity and magnetic permeability, respectively.

In the language of differential forms they can be written as

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- together with the constitutive relations

$$* \, B = \mu H \qquad * \, D = \varepsilon E, \tag{11}$$

  where ε and μ are the operators induced by $[\varepsilon], [\mu]$ on 1-forms, respectively.

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- Then the 1-form $\alpha = *D \in \Omega^1(\mathcal{U})$ satisfies the differential equation

$$\delta M d N \alpha - \omega^2 \alpha = 0, \quad \delta\alpha = 0 \tag{12}$$

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- where $(\det[\mu]) \cdot M$ is the operator induced by $[\mu]$ on 2-forms, and N is the operator induced by $[\varepsilon]^{-1}$ on 1-forms.

A standard approach to the Dirichlet problem with relative boundary conditions in a bounded open set $\mathcal{U}$ for system (12) relies on a variational argument for the functional

$$\begin{aligned}\tilde{J}^{\mu,\varepsilon}(\alpha) := \int_{\mathcal{U}} \langle M dN\alpha, dN\alpha\rangle_{\mathrm{Euc}}\, dV + \sigma \int_{\mathcal{U}} |\delta\alpha|^2\, dV \\ + C \int_{\mathcal{U}} \langle N\alpha, \alpha\rangle_{\mathrm{Euc}}\, dV.\end{aligned} \tag{13}$$

Here, $\sigma > 0$ is a positive parameter, and $C > 0$ is a large constant.

Mimicking the Euclidean approach, in $\mathbb{H}^1$ we have to consider the functional

$$J(\alpha) := \int_{\mathcal{U}} |d_c\alpha|^2\, dV + \sigma \int_{\mathcal{U}} |\delta_c\alpha|^2\, dV + C \int_{\mathcal{U}} |\alpha|^2\, dV,$$

which still hides all the peculiarities of the structure of our intrinsic Maxwell's equations.

To prove that Maxwell's equations in the groups are limits of Maxwell's equations in very anisotropic media, we show that the functional J is the Γ-limit of functionals $\tilde{J}^{\mu,\varepsilon}$ for suitable choices of $[\mu], [\varepsilon]$.

Definition

Let X be a metric space, and let

$$J_r,\ J:\ X \longrightarrow [-\infty, +\infty]$$

with $r > 0$ be functionals on X. Then $\{J_r\}_{r>0}$ Γ-converges to J on X as r goes to zero if and only if the following two conditions hold:

- for every $u \in X$ and for every sequence $\{u_{r_k}\}_{k\in\mathbb{N}}$ with $r_k \to 0$ as $k \to \infty$, which converges to u in X, there holds

$$\liminf_{k\to\infty} J_{r_k}(u_{r_k}) \geq J(u); \tag{14}$$

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- for every $u \in X$ and for every sequence $\{r_k\}_{k\in\mathbb{N}}$ with $r_k \to 0$ as $k \to \infty$ there exists a subsequence (still denoted by $\{r_k\}_{k\in\mathbb{N}}$) such that $\{u_{r_k}\}_{k\in\mathbb{N}}$ converges to u in X and

$$\limsup_{k\to\infty} J_{r_k}(u_{r_k}) \leq J(u) \tag{15}$$

If $r > 0$, consider now the functional

$$J_r := \tilde{J}^{\mu,\varepsilon},$$

where the magnetic permeability $[\mu_r]$ and dielettric permittivity $[\varepsilon_r]$ have the forms

$$[\mu_r] = [\varepsilon_r] = \begin{pmatrix} 1 + \frac{r}{4}y^2 & -\frac{r}{4}xy & \frac{r}{2}y \\ -\frac{r}{4}xy & 1 + \frac{r}{4}x^2 & -\frac{r}{2}x \\ \frac{r}{2}y & -\frac{r}{2}x & r \end{pmatrix}.$$

Definition

If $\alpha \in \Omega^1(\mathcal{U})$, we write

$$\alpha = \alpha_1 + \alpha_2,$$

with $\alpha_i \in \Omega^{1,i}(\mathcal{U})$, $i = 1, 2$. If $m \geq 2$, we say that

$$\alpha \in \widehat{W}^{m,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g}) \quad \text{iff} \quad \alpha_i \in W^{m+1-i,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g}), \quad i = 1, 2.$$

The space $\widehat{W}^{m,2}_{\mathbb{G}}(\mathcal{U}, \bigwedge^1 \mathfrak{g})$ is endowed with its natural norm.

Theorem (Baldi, Franchi 2011)

Let Ω be a bounded open set in $\mathbb{H}^1$ with smooth boundary. The restriction to $W^{3,2}_{\mathbb{H}^1}(\Omega, \bigwedge^1 \mathfrak{g})$ of the functional J_r Γ-converges, rwith respect to the topology induced by $W^{2,2}_{\mathbb{H}^1}(\Omega, \bigwedge^1 \mathfrak{g})$, to the restriction to $W^{3,2}_{\mathbb{H}^1}(\Omega, \bigwedge^1 \mathfrak{g})$ of the functional

$$J(\alpha) = \begin{cases} \displaystyle\int_{\mathbb{R}^3} |d_c\alpha|^2\, dV + \int_{\mathbb{R}^3} |\delta_c\alpha|^2\, dV + C \int_{\mathbb{R}^3} |\alpha|^2\, dV \\ \qquad\qquad\qquad \text{se } \alpha \in W^{2,2}_{\mathbb{H}^1}(\mathbb{H}^1, \bigwedge^1 \mathfrak{g}) \\ \qquad\qquad\qquad + \text{ suitable boundary conditions,} \\ +\infty \quad \text{otherwise.} \end{cases}$$