Differential-geometric and invariance properties of the equations of MP

I would appreciate if my talk is considered not as a report on some recent results in optimal control, but rather as a lecture on its foundations, a modest offering to the jubilee of Andrey's 60–th anniversary, and hopefully, for a possible use in his future book on Sub-Riemannian geometry.

1. MP and the Pontryagin derivative $\mathcal{P}_X$.

The time-optimal problem and invariance of its formulation:

$$\frac{dx}{dt} = X(x, u), \;\; x \in M, \;\; u \in U;$$

$$x(t), u(t), \;\; t \in J,$$

$$\frac{d}{dt} x(t) = X(x(t), u(t)), \; u(t) \in U,$$

$$x(t_1) \longrightarrow x(t_2) \Longrightarrow t_2 - t_1 = min,$$

$$\forall t, t_1, t_2 \in J.$$

The problem is completely defined by the family of vector fields $X \in Vect\, M$.

Hence our goal consists in representing the equations of MP as canonical invariants of the vector field $X \in Vect\, M$.

Preliminary constructions:

$$\psi = (\psi_1, \ldots, \psi_n),$$
$$(O, x),\ O \subset M, \quad X = \sum_\alpha X^\alpha \frac{\partial}{\partial x^\alpha}.$$

The *Hamiltonian lift* (in a noninvariant form, depending on the choice of the neighborhood (O, x)),

$$\left.\begin{aligned} X = \sum_\alpha X^\alpha \frac{\partial}{\partial x^\alpha} \mapsto H = \\ \sum_\alpha \psi_\alpha X^\alpha(x, u) \mapsto \mathcal{P}_X, \\ \mathcal{P}_X = \sum_\alpha \frac{\partial H}{\partial \psi_\alpha}\frac{\partial}{\partial x^\alpha} - \sum_\alpha \frac{\partial H}{\partial x^\alpha}\frac{\partial}{\partial \psi_\alpha}. \end{aligned}\right\} \quad (1)$$

The *maximum condition* (for parameter elimination),

$$H(\psi, x, u) = \max_v H(\psi, x, v).$$

MP asserts:

Nontrivial extremals of the problem,

$$(\psi(t), x(t)), \psi(t) \neq 0,\ t \in J,$$

are trajectories of the Hamiltonian field $\mathcal{P}_X$ (with parameter u) generated as a result of *"dynamical elimination"* of the parameter by the maximum condition, as we proceed along the trajectory. Every solution to the time-optimal problem is represented as a projection onto the x-space of a nontrivial extremal.

Thus the *"Pontryagin derivative"* $\mathcal{P}_X$ should be considered as the main "variational derivative" of the problem that contains, together with the maximum condition, complete first order information about the time-optimal problem.

Our goal consists:

In presenting the sequence (1),

$$X = \sum_\alpha X^\alpha \frac{\partial}{\partial x^\alpha} \mapsto H = \sum_\alpha \psi_\alpha X^\alpha(x,u) \mapsto \mathcal{P}_X,$$

as a canonically invariant construction and expressing the Pontryagin derivative $\mathcal{P}_X$ as a natural $\mathbb{R}$-linear correspondence — a *"Hamiltonian lift to T^*M over the vector field $X \in Vect\, M$"*,

$$X \mapsto \mathcal{P}_X \in Vect\, T^*M, \;\; P_{\lambda X+\mu Y} = \lambda \mathcal{P}_X + \mu \mathcal{P}_Y,$$

from which the basic properties of $\mathcal{P}_X$ automatically follow. This also provides *an invariant representation of MP.*

Finally, we shall identify the Hamiltonian vector field $\mathcal{P}_X$ with basic differential-geometric objects on M.

2. <u>The Hamiltonian lift</u> :

$$X \mapsto \mathcal{P}_X \in \mathfrak{H}(T^*M) \subset Vect\,T^*M$$

The $\mathbb{R}$-algebras $C^\infty(T^*M), C^\infty(TM)$ as modules over the “ring of scalars” $C^\infty(M)$:

$$T^*M \xrightarrow{\pi} M,\ TM \xrightarrow{pr} M;$$

$$aH' + bH'' \stackrel{def}{=} \pi^*a \cdot H' + \pi^*b \cdot H'',$$

$$\forall a, b \in C^\infty(M), H', H'' \in C^\infty(T^*M).$$

Basic submodules of $C^\infty(T^*M), C^\infty(TM)$:

$$\mathfrak{Q}, \mathfrak{P} \subset C^\infty(T^*M), \quad \mathcal{Q}, \dot{\mathcal{Q}} \subset C^\infty(TM);$$

$$\mathfrak{Q} = \left\{ q = \pi^*a \,\middle|\, a \in C^\infty(M) \right\},$$

$$\mathcal{Q} = \left\{ q = pr^*a \,\middle|\, a \in C^\infty(M) \right\};$$

$$\mathfrak{P} = \left\{ p \in C^\infty(T^*M) \,\middle|\, p\big|_{T_x^*M} \in L(T_x^*M, \mathbb{R})\ \forall x \in M \right\},$$

$$\dot{\mathfrak{Q}} = \left\{ \dot{q} \in C^\infty(TM) \,\middle|\, \dot{q}\big|_{T_xM} \in L(T_xM, \mathbb{R})\ \ \forall x \in M \right\}.$$

Canonical identifications:

$$\mathfrak{P} \stackrel{\sim}{=} Vect\,M,\ \dot{\mathcal{Q}} \stackrel{\sim}{=} \Lambda^1 M.$$

Canonical embedding of $Vect\,M$ into $\mathfrak{P}$:

$$Vect\,M \subset \mathfrak{P} \subset C^\infty(T^*M),\;\; Z : x \mapsto Z_x \in T_xM,$$
$$H_Z(\sigma) = < \sigma, Z_{\pi\sigma} > \;\; \forall \sigma \in T^*_xM, x \in M.$$

Hamiltonian lift to T^*M over $Z \in Vect\,M$:

$$Z \mapsto H_Z \mapsto D_Z, \quad i_{D_Z}\omega = -dH_Z,$$
$$Z \in Vect\,M,\;\; H_Z \in \mathfrak{P},\;\; D_Z \in \mathfrak{H},$$
$$Vect\,M \longrightarrow \mathfrak{H} \subset Vect\,T^*M\; :$$

D_Z preserves the submodule $\mathfrak{P} \subset C^\infty(T^*M)$ and $D_Z\Big|_{\mathfrak{P}}$, is a derivation over Z of the $C^\infty(M)$-module $\mathfrak{P}$, i.e. it is $\mathbb{R}$-linear and satisfies the Leibniz identity,

$$D_Z \cdot ap = Za \cdot p + a(D_Z\,p) \quad \forall a \in C^\infty(M),\;\; p \in \mathfrak{P}$$

$\Downarrow$

the corresponding flow e^{tD_Z} on T^*M is a lift over the flow e^{tZ},

$$e^{tD_Z}\Big|_{T^*_xM} : T^*_xM \longrightarrow T^*_{e^{tX}x}M.$$

3. Invariant representation of $\mathcal{P}_X$

Apply the *Hamiltonian lift* to the family $X \in Vect\, M$ (for every fixed u), then

$$X \mapsto H_X \mapsto \mathcal{P}_X, \;\; i_{\mathcal{P}_X}\omega = -dH_X,$$
$$X \in Vect\, M, \;\; H_X \in \mathfrak{P}, \;\; \mathcal{P}_X \in \mathfrak{H}.$$

Indeed, in canonical coordinates (q, p) on T^*M over an arbitrary coord. nghb. $(O, x), O \subset M$, the Hamiltonian $H_X \in \mathfrak{P}_X$ has the expression,

$$H = \sum_\alpha p_\alpha X^\alpha = pX,$$

$$\left(\pi^{-1}U, (q, p)\right), \;\; q = \pi^* x, \; p = \frac{\partial}{\partial x}, \;\; U \subset M,$$

$$H_X(\sigma) = < \sigma, X\Big|_{\pi\sigma} > =$$

$$\sum_{\alpha,\beta} < p_\alpha(\sigma) dx^\alpha\Big|_{\pi\sigma}, X^\beta \frac{\partial}{\partial x^\beta}\Big|_{\pi\sigma} > =$$

$$\sum_\alpha p_\alpha(\sigma) X^\alpha(\pi\sigma) = \sum_\alpha p_\alpha X^\alpha\Big|_\sigma \;\; \forall \sigma \in T^*M.$$

Explicit expression for $\mathcal{P}_X$ in canonical coordinates

$$\mathcal{P}_X = \frac{\partial(pX)}{\partial p}\frac{\partial}{\partial q} - \frac{\partial(pX)}{\partial q}\frac{\partial}{\partial p} = \;\; X\frac{\partial}{\partial q} - p\frac{\partial X}{\partial x}\frac{\partial}{\partial p}$$

4. The Lie derivative $\mathcal{L}_X$.

A standard lift to the tangent bundle TM over X of "every-day usage" is the *Lie derivative* $\mathcal{L}_X \in Vect\,TM$:

$$\left(e^{tX}\right)_* = e^{t\mathcal{L}_X}, \;\; e^{t\mathcal{L}_X}\Big|_{T_xM} : T_xM \longrightarrow T_{e^{tX}x}M.$$

The *adjoint flow* $\left(e^{t\mathcal{L}_X}\right)^{\#} \stackrel{def}{=} e^{t\mathcal{L}^{\#}_X}$ *is a lift to* T^*M *over* e^{-tX}*:*

$$\left.\begin{array}{c} < e^{t\mathcal{L}^{\#}_X}\theta_x, Y_{e^{-tX}x} > = < \theta_x, e^{t\mathcal{L}_X} Y_{e^{-tX}x} > \\ \forall \theta_x \in T^*_xM, \;\; Y_{e^{-tX}x} \in T_{e^{-tX}x}M. \end{array}\right\} \quad (2)$$

The *dual flow to* $e^{t\mathcal{L}_X}$,

$$\left(e^{t\mathcal{L}^{\#}_X}\right)^{-1} = e^{-t\mathcal{L}^{\#}_X} = e^{t\mathcal{L}^{\#}_{-X}},$$

— a lift to T^*M over e^{tX}. Respectively, the *dual field* $\mathcal{L}^{\#}_{-X}$ *to the Lie derivative* $\mathcal{L}_X$ *is a lift to* T^*M *over* X*.*

Our basic conjecture — *is Pontryagin derivative identical with the dual to the Lie derivative?*

$$\boxed{\mathcal{P}_X = \mathcal{L}^{\#}_{-X}} \quad ?$$

5. Identification of the Pontryagin derivative.

Computing explicitly the dual $\mathcal{L}^{\#}{}_{-X}$ requires to consider relation (2) between the adjoint flows of diffeomorphisms as a relation between the corresponding flows of algebra automorphisms, which is easily expressed as

$$< exp(t\mathcal{L}_X)\theta, Y >= exp(tX) < \theta, exp(t\mathcal{L}^{\#}_X)Y > .$$

Differentiating the obtained expression with resp. to t and then putting $t = 0$ yields,

$$X < \theta, Y >=< \mathcal{L}_X\theta, Y > + < \theta, \mathcal{L}^{\#}{}_{-X}Y >,$$
$$\forall\theta \in \Lambda^{(1)}(M),\ \ X, Y \in Vect\, M$$
$$\Updownarrow$$
$$X < \theta, Y >=< \mathcal{L}_X\theta, Y > + < \theta, ad_X Y >,$$
$$\Downarrow$$
$$\mathcal{L}^{\#}{}_{-X} = ad_X \stackrel{?}{=} \mathcal{P}_X.$$

This was an unexpected but easily verifiable conjecture since a short computation yields,

$$\mathcal{P}_X H_Y = H_{ad_X Y} = H_{[X,Y]} \Rightarrow \mathcal{P}_X\Big|_{\mathfrak{P}} = ad_X,$$

hence $\mathcal{P}_X$ *is a Hamiltonian extension of* ad_X.

6. Formulation of the final result:

The Pontryagin derivative $\mathcal{P}_X$ is a Hamiltonian lift to T^*M over the vector field $X \in Vect\,M$, invariantly derived from X by the canonical sequence,

$$X \mapsto H_X \in \mathfrak{P} \subset C^\infty(T^*M) \mapsto \mathcal{P}_X,$$
$$i_{\mathcal{P}_X}\omega = -dH_X,$$

and the natural correspondence $X \mapsto \mathcal{P}_X$ is $\mathbb{R}$-linear. Hence the corresponding flow $e^{t\mathcal{P}_X}$ is a Hamiltonian lift to T^*M over e^{tX}.
The Hamiltonian vector field $\mathcal{P}_X$ is dual to the Lie derivative $\mathcal{L}_X \in Vect\,TM$ and is an extension of the Lie bracket ad_X, where ad_X is considered as a derivation over X on the submodule $\mathfrak{P} \subset C^\infty(T^*M)$ of fiberwise linear functions (vector fields on M).

Thus the whole computational power of MP is based, via the "dynamical elimination procedure", on two rudimentary differential-geometric notions — the Lie derivative, (the infinitesimal variations of the optimal system), and its dual — the Pontryagin derivative, which, in fact, coincides with the Lie bracket.

7. Two remarks.

I would like to finish my talk with two final remarks, the first of which concerns Hamiltonians of variational problems, as they are presented in Physics and Mechanics texts, where the Hamiltonians are considered rather as sacred objects, expressing the energy of the system. The authors always try to give more or less convincing physical arguments that could predict at least the general properties of the Hamiltonian for the variational problem under consideration, whereas in mathematics texts one can never find similar, but purely mathematical, arguments. One can easily give such arguments, if we consider the invariant representation of MP described above. Indeed, if the initial optimal problem is canonically reduced to a time-optimal problem with the corresponding vector field X on the configuration space M, then the Hamiltonian of the initial problem is uniquely defined as the vector field X itself,

$$X \overset{\sim}{=} H_X \in \mathfrak{P} \subset C^\infty(T^*M).$$

My second remark concerns the time-optimal problem with restricted phase coordinates. The problem was already considered in our book in 1962, in form of a MP with some modifications, which took into account the motion on the boundary of the region. Since then, the problem was reconsidered several times, the latest publications, as far as I know, appearing quite recently. The search always was towards the most simple and perfect analog of MP. And as always in such situations, every author considered his own version as the most perfect. I think, if we consider the problem from the viewpoint adopted in this talk, and succeed in finding an invariant formulation of the corresponding MP, the result should definitely be the best possible, regardless of tastes of individual authors. It seems that the only, though essential, difference from our considerations should consist in the assumption that the configuration space of the problem M is not a closed manifold, but rather a manifold with boundary.

Thank you for attention!