A new class of $(\mathcal{H}^\alpha, 1)$-rectifiable subsets of metric spaces

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Cortona, May 2012

Outline

Original motivations

Computation of **Hausdorff measures**

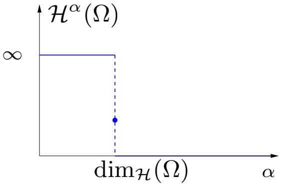
in **Carnot-Carathéodory spaces**

Hausdorff measures

(M,d) metric space – $\Omega \subset M$ – $\alpha \geq 0$ real number

$$\mathcal{H}^\alpha(\Omega) = \lim_{\delta\to 0} \inf\{\sum_i (\text{ diam } F_i)^\alpha \; : \; \Omega \subset \cup_i F_i, \quad F_i \text{ closed, diam } F_i < \delta\}$$

$$\mathcal{S}^\alpha(\Omega) = \lim_{\delta\to 0} \inf\{\sum_i (\text{ diam } F_i)^\alpha \; : \; \Omega \subset \cup_i F_i, \quad F_i \text{ closed ball, diam } F_i < \delta\}$$



$$\mathcal{H}^\alpha(\Omega) \leq \mathcal{S}^\alpha(\Omega) \leq 2^\alpha \mathcal{H}^\alpha(\Omega)$$

Carnot–Carathéodory spaces

Definition

A **sub-Riemannian manifold** is a triple (M, D, g) where

- M smooth (i.e., $\mathcal{C}^\infty$) manifold
- $D \subset TM$ subbundle
- $g(\cdot,\cdot)$ Riemannian metric on D

horizontal path $\rightarrow$ $\gamma : [a, b] \rightarrow M$

- γ absolutely continuous
- $\dot\gamma(t) \in D(\gamma(t))$ a.e.

length $\rightarrow$ $\mathrm{length}(\gamma) = \int_a^b \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt$

γ non-horizontal $\Rightarrow$ $\mathrm{length}(\gamma) = +\infty$

sub-Riemannian distance

$$d(p,q) = \inf\{\mathrm{length}(\gamma) \ : \ \gamma \text{ horizontal joining } p \text{ to } q\}$$

D is non-holonomic, i.e., for every $p \in M$ there exists $r(p) \in \mathbb{N}$ s.t.

$$\{0\} \subset D^1(p) \subset D^2(p) \subset \cdots \subset D^{r(p)}(p) = T_pM.$$

Theorem (Chow-Rashevsky)

M connected, D non-holonomic $\Rightarrow$ $d(p,q) < \infty$ and d is continuous.

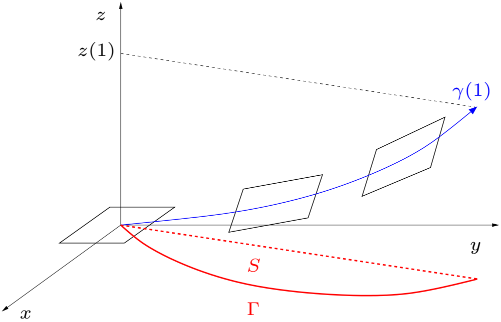
The metric space (M,d) is called **Carnot–Carathéodory space**.

An example: the Heisenberg group

$$M = \mathbb{R}^3,\ D = \ker\left(dz - \frac{1}{2}(ydx - xdy)\right),\ g = dx^2 + dy^2$$

$z(1) =$ area S enclosed

by the projection Γ



$\rightarrow$ Distance restricted to the z-axis:

$$d\big((0,0,z),(0,0,z')\big) = 2\sqrt{\pi}\sqrt{|z - z'|}.$$

Let C = interval $[0, z]$ on the z-axis. Then

$$\mathcal{H}^2(C) = 4\pi z \qquad (\text{or } \mathcal{H}^2|_C = 4\pi\mathcal{L}^1)$$

and the 2-density of C is

$$\lim_{r\to 0^+} \frac{\mathcal{H}^2(C \cap B(q,r))}{r^2} = 2.$$

This raises two questions:

- integral representations for $\mathcal{H}^\alpha(C)$? Comparison with other measures?
- relationships between α-density and rectifiability?

Measures in SR manifolds : existing notions and results

- top dimensional spherical Hausdorff measure

  $\rightarrow$ *[Agrachev - Barilari - Boscain - Gauthier]*

For paths:

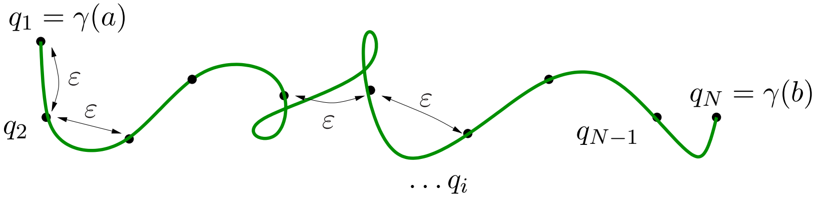
- $\mathcal{H}^1$ has an integral formulation (for absolutely continuous paths):

$$\mathcal{H}^1(\gamma) = \mathrm{length}(\gamma) = \int \sqrt{g(\dot\gamma(t),\dot\gamma(t))}dt$$

- $\exists$ other notions of integral measures $\rightarrow$ *[Falbel - J. '03]*
  but not related to Hausdorff measures up to now

- approximation by finite sets: complexity $\sigma(\cdot, \varepsilon)$

  $\gamma : [a, b] \to M$ continuous path, $C = \gamma([a, b])$, $\varepsilon > 0$



$$\sigma(C, \varepsilon) = \min \{N \mid \exists\ q_1 = \gamma(a), \dots, q_N = \gamma(b),\ d(q_i, q_{i+1}) \leq \varepsilon\}$$

There exists equivalents in integral form

[J., Gauthier - Zakalyukin '00s]

Rectifiability theory: classical definition

Definition

$$(M, d) \text{ metric space} \quad - \quad k \in \mathbb{N}$$

$S \subset M$ is $(\mathcal{H}^k, k)$-**rectifiable** if there exists a countable family of Lipschitz functions $f_i : \mathbb{R}^k \to M$ such that

$$\mathcal{H}^k(S \setminus \bigcup_i f_i(\mathbb{R}^k)) = 0.$$

[Federer 1947] in the Euclidean space, *[Kirchheim 1994]*

Properties in the Euclidean space:

- Rademacher Theorem $\Rightarrow$ Lipschitz can be replaced by $\mathcal{C}^1$
- S is $(\mathcal{H}^k, k)$-rectifiable $\Leftrightarrow$ S included in the union of a set of zero $\mathcal{H}^k$-measure and a countable family of $\mathcal{C}^1$ manifold of dimension k
- $C = \gamma([a, b])$ is $(\mathcal{H}^1, 1)$-rectifiable $\Leftrightarrow$ C has finite length

Theorem (Federer, Kirchheim)

Assume S is $\mathcal{H}^k$-measurable and $\mathcal{H}^k(S) < \infty$.
Then S is $(\mathcal{H}^k, k)$-rectifiable $\Rightarrow$

$$0 < \lim_{r \to 0} \frac{\mathcal{H}^k(S \cap B(x,r))}{r^k} < \infty, \quad \textit{for } \mathcal{H}^k\textit{-a.e. } x \in S$$

When (M, d) = Euclidean space, "$\Rightarrow$" becomes "$\Leftrightarrow$" *[Preiss 1987]*

In a Carnot-Carathéodory space:

Lipschitz paths $\Leftrightarrow$ horizontal paths

$\Rightarrow$ $C = \gamma([a,b])$ is $(\mathcal{H}^1, 1)$-rectifiable iff γ horizontal
$\Rightarrow$ <u>smooth</u> non-horizontal paths are not 1-rectifiable ...

$+$ in Heisenberg: z-axis not 2-rectifiable but has a constant 2-density

Questions and strategy

1. How to compute Hausdorff measure of non-horizontal paths?
2. How to define 1-rectifiable sets in a such a way that smooth non-horizontal paths are rectifiable?

$\rightarrow$ **new class of curves** (in any metric space)

- allows to compute measures on continuous curves
- can be used to define rectifiable sets with suitable properties

Outline

m-$\mathcal{C}^1_\alpha$ paths and infinitesimal measures

(M, d) metric space

Definition

$\gamma : [a, b] \to M$ continuous path, $\alpha \geq 1$ real number.

γ is **continuously metric differentiable of degree** α **at** t ($=$ **m-**$\mathcal{C}^1_\alpha$) if

$$\lim_{s \to 0} \frac{d(\gamma(t+s), \gamma(t))}{|s|^{1/\alpha}}$$

exists and depends continuously on t.

The α-**measure of** γ **at** t is

$$\mathrm{meas}^\alpha_t(\gamma) = \begin{cases} \left(\lim_{s \to 0} \frac{d(\gamma(t+s), \gamma(t))}{|s|^{1/\alpha}} \right)^\alpha & \text{if } \gamma \in \text{m-}\mathcal{C}^1_\alpha \\ +\infty & \text{otherwise.} \end{cases}$$

The classical case: $\alpha = 1$

- $\lim_{s\to 0} \frac{d(\gamma(t+s),\gamma(t))}{|s|}$ $\to$ **metric derivative** of γ at t
- m-$\mathcal{C}^1_1$ $\Rightarrow$ Lipschitz
- (M, d) CC space $\Rightarrow$

$$\lim_{s\to 0} \frac{d(\gamma(t+s),\gamma(t))}{|s|} = \sqrt{g(\dot\gamma(t),\dot\gamma(t)} = \mathrm{meas}^1_t(\gamma),$$

at each point t where γ is differentiable

Integral measures: α-lengths

Definition

$\gamma : [a, b] \to M$ continuous, $C = \gamma([a, b])$, $\alpha \geq 1$ real number.

$$\mathrm{Length}_\alpha(C) = \int_a^b \mathrm{meas}_t^\alpha(\gamma)\, dt.$$

(M, d) CC space and $\alpha = 1 \;\Rightarrow\; \mathrm{length}(\gamma) = \mathrm{Length}_1(C)$.

Theorem (Ambrosio, Kirchheim '90s)

γ *injective, Lipschitz* $\;\Rightarrow\;$ *the metric derivative of* γ *exists a.e. and*

$$\mathcal{H}^1(\gamma([a, b])) = \int_a^b \mathrm{meas}_t^1(\gamma)\, dt \qquad [= \mathrm{length}(\gamma),\ \textit{in a CC space}].$$

Examples in CC spaces

- $\gamma \in \mathcal{C}^1$, γ horizontal $\Rightarrow$ $\gamma \in$ m-$\mathcal{C}^1_1$ and $\mathrm{meas}^1_t(\gamma) \neq 0$
- the z-axis in Heisenberg is m-$\mathcal{C}^1_2$ with $\mathrm{meas}^2_t = 4\pi$

Definition

$\gamma : [a,b] \to M, \ C = \gamma([a,b])$

- γ is C-regular at t if $\dim D^k(\gamma(s))$ is constant near t, for $k \in \mathbb{N}$.
- γ is equiregular if it is C-regular at any $t \in [a,b]$

Proposition

$$\left.\begin{array}{c} \gamma \in \mathcal{C}^1, \ \textit{equiregular} \\ \dot\gamma(t) \in D^k(\gamma(t)) \end{array}\right\} \Rightarrow \gamma \in m\text{-}\mathcal{C}^1_k$$

$$\dot\gamma(t) \notin D^{k-1}(\gamma(t)) \ \Rightarrow \ \mathrm{meas}^k_t(\gamma) \neq 0.$$

- **Remark:** m-$\mathcal{C}^1_\alpha$ $\not\Rightarrow$ $\mathcal{C}^1$

In a Riemannian manifold, i.e., $D = TM$,

Proposition (Riemannian case)

$$\left.\begin{array}{c} \gamma \in \ m\text{-}\mathcal{C}^1_\alpha \\ \operatorname{meas}^\alpha_t(\gamma) \not\equiv 0 \end{array}\right\} \Rightarrow \ \alpha = 1$$

$\rightarrow$ all non trivial m-$\mathcal{C}^1_\alpha$curves are Lipschitz

Properties of m-$\mathcal{C}^1_\alpha$ paths

- **Hölder continuity**: $\gamma \in$ m-$\mathcal{C}^1_\alpha \Rightarrow \gamma$ is Hölder of order $1/\alpha$.
  If moreover $\mathrm{meas}^\alpha_t(\gamma)$ never vanishes, γ is bi-Hölder :

$$k_1|s|^{1/\alpha} \leq d(\gamma(t+s), \gamma(t)) \leq k_2|s|^{1/\alpha}.$$

Theorem

γ injective, m-$\mathcal{C}^1_\alpha$, $C = \gamma([a,b]) \;\Rightarrow$

$$\mathcal{H}^\alpha(C) = \mathcal{S}^\alpha(C) = \mathrm{Length}_\alpha(C) = \lim_{\varepsilon\to 0} \varepsilon^\alpha \sigma(C,\varepsilon),$$

$$\text{and} \quad \lim_{r\to 0^+} \frac{\mathcal{H}^\alpha(C\cap B(q,r))}{r^\alpha} = 2 \quad \text{for } \mathcal{H}^\alpha\text{-a.a. } q \in C.$$

$$\Rightarrow \qquad \frac{d\mathcal{H}^\alpha\lfloor_C}{d\gamma_*\mathcal{L}^1}(t) = \mathrm{meas}^\alpha_t(\gamma)$$

Consequences in Carnot-Carathéodory spaces:

- Computations of $\mathcal{H}^\alpha(C)$ for smooth paths through $\sigma(C,\varepsilon)$
- Define:

$$\begin{aligned} m_C &= \inf\{k \in \mathbb{N} \ : \ \dot\gamma(t) \in D^k(\gamma(t)) \text{ for a.e. } t\} \\ I_C &= \{t \in [a,b] \ : \ \gamma \text{ is not } \mathcal{C}^1 \text{ or } \gamma \text{ is } C-\text{singular at } t\}. \end{aligned}$$

Theorem

If γ injective and continuous, I_C closed and $\mathcal{L}^1(I_C) = \mathcal{H}^{m_C}(\gamma(I_C)) = 0$, then

$$\mathcal{H}^\alpha(C) = \mathcal{S}^\alpha(C) = \mathrm{Length}_\alpha(C) \qquad \textit{for any } \alpha \geq 1,$$

$$\textit{and} \qquad \dim_{\mathcal{H}} C = m_C.$$

Outline

1-rectifiability: new definition

(M, d) metric space

Definition

A set $S \subset M$ is $(\mathcal{H}^\alpha, 1)$-**rectifiable** if there exists a countable family of m-$\mathcal{C}^1_\alpha$ paths $\gamma_i : I_i \to M$, I_i closed intervals, such that

$$\mathcal{H}^\alpha(S \setminus \bigcup_{i \in \mathbb{N}} \gamma_i(I_i)) = 0.$$

Remark

If (M, d) CC space, then:

$$\left.\begin{array}{c} k \in \mathbb{N},\ \gamma\ \mathcal{C}^1 \textit{ curve} \\ \gamma \textit{ equiregular} \\ \dot\gamma(t) \in D^k(\gamma(t)) \textit{ for every } t \end{array}\right\} \Rightarrow \gamma([a,b]) \textit{ is } (\mathcal{H}^k, 1)\textit{-rectifiable}$$

A density result

Theorem

Let $S \subset M$ be $\mathcal{H}^\alpha$-measurable with $\mathcal{H}^\alpha(S) < +\infty$.
If S is $(\mathcal{H}^\alpha, 1)$-rectifiable then, for $\mathcal{H}^\alpha$-almost every $q \in S$,

$$2 \leq \liminf_{r \to 0^+} \frac{\mathcal{H}^\alpha(S \cap B(q,r))}{r^\alpha} \leq \limsup_{r \to 0^+} \frac{\mathcal{H}^\alpha(S \cap B(q,r))}{r^\alpha} \leq 2^\alpha.$$

- no linear structure
- measures computations and density property of m-$\mathcal{C}^1_\alpha$ paths

To be compared with:

Theorem (Preiss 1987)

$\exists c > 1$ s.t., for $S \in \mathbb{R}^n$ and $k \in \mathbb{N}$: for $\mathcal{H}^k$-a.e. $q \in S$

$$0 < \limsup_{r \to 0^+} \frac{\mathcal{H}^k(S \cap B(q,r))}{r^k} \leq c \liminf_{r \to 0^+} \frac{\mathcal{H}^k(S \cap B(q,r))}{r^k},$$

if and only if S is $(\mathcal{H}^k, k)$-rectifiable.

Outline

Open problems I

In the Theorem of Preiss, the constant c is close to 1. In our result, the corresponding constant is $2^{\alpha-1}$.

Conjecture

The α-density exists,

$$\lim_{r\to 0^+} \frac{\mathcal{H}^\alpha(S\cap B(q,r))}{r^\alpha} = 2 \qquad \mathcal{H}^\alpha - \text{a.e.}$$

Far more difficult problem: does a reciprocal of our Theorem exists?

Open problems II

In SR geometry, there exists a notion of **metric differential of degree** α **of** γ **at** t, denoted α-$d\gamma_t$. *[J. - Falbel]*

α-$d\gamma_t$ exists $\forall t$ and is continuous $\Rightarrow$ γ is m-$\mathcal{C}^1_\alpha$

Question

$$\gamma \text{ m-}\mathcal{C}^1_\alpha \quad \Rightarrow \quad \alpha\text{-}d\gamma_t \text{ exists for a.e. } t \text{ ?}$$

(Rademacher's like theorem)

Remark: γ m-$\mathcal{C}^1_\alpha$ + $\mathrm{meas}^\alpha_t(\gamma)$ nontrivial $\Rightarrow \gamma$ bi-Hölder of degree $1/\alpha$

Is it possible to replace m-$\mathcal{C}^1_\alpha$ curves with bi-Hölder ones in the preceding constructions?

NO, because in general:

- no density results holds for bi-Hölder curves;
- γ bi-Hölder of degree $1/\alpha \quad \not\Rightarrow \quad \alpha$-$d\gamma_t$ exists for a.e. t

Consequence:
m-$\mathcal{C}^1_\alpha$ curves and $(\mathcal{H}^k, 1)$-rectifiable sets are not preserved by bi-Lipschitz equivalence

Conjecture

Bi-Lipschitz maps between CC-spaces do not preserve smooth non-horizontal maps

Open problems III

(M, D, g) equiregular SR manifold
$k \in \{1, \ldots, r(p)\} \;\Rightarrow\;$ there exists γ m-$\mathcal{C}^1_k$ with $\mathrm{meas}^k_t(\gamma) \not\equiv 0$.

Proposition

γ absolutely continuous and m-$\mathcal{C}^1_\alpha$ with $\mathrm{meas}^\alpha_t(\gamma) \not\equiv 0 \;\Rightarrow\; \alpha \in \mathbb{N}$.

Question: What if γ is not absolutely continuous?

To be compared with Marstrand's theorem in the Euclidean space:

Theorem (Marstrand 1964)

Let $\alpha \geq 0$ s.t. there exists $B \subset \mathbb{R}^n$ satisfying $\mathcal{H}^\alpha(B) > 0$ and

$$0 < \lim_{r \to 0} \frac{\mathcal{H}^\alpha(B \cap B(x, r))}{r^\alpha} < +\infty, \quad \forall x \in B.$$

Then $\alpha \in \mathbb{N}$.