Jacobi's geodesic problem and Lie groups

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Jacobi's problem

- Jacobi's Geodesic problem-1835: Find a curve $x(t)$ on the ellipsoid $E=\{x\in R^{n+1}:\sum_{i=0}^{n}\frac{1}{a_i}x_i^2=1$ that connects two given points of E whose length $L=\int_0^1\sqrt{\sum_{i=0}^{n}(\frac{dx_i}{dt})^2}\,dt$ is minimal.
- Jacobi's method of solution: Elliptical coordinates: u_0,u_1,u_2. Solutions of $\frac{x_1^2}{a_1-u}+\frac{x_2^2}{a_2-u}+\frac{x_3^2}{a_3-u}=1$
- If $a_1<a_2<a_3$, then $u_0<a_1<u_1<a_2<u_2<a_3$.
- u_1,u_2 are the coordinates on the ellipsoid $\sum_{i=1}^{3}\frac{x_i^2}{a_i-u_0}=1$.
- $ds^2=(u_2-u_1)(\frac{u_2-u_0}{f(u_2)}du_2^2-\frac{u_1-u_0}{f(u_1)}du_1^2), f(u)=4(a_1-u)(a_2-u)(a_3-u)$
- Associated Hamiltonian: $H=\frac{1}{u_2-u_1}(\frac{f(u_2)}{u_2-u_0}p_2^2-\frac{f(u_1)}{u_1-u_0}p_1^2)=1.$
- $\frac{1}{u_2-u_1}(\frac{f(u_2)}{u_2-u_0}\frac{\partial S}{\partial u_2}^2-\frac{f(u_1)}{u_1-u_0}\frac{\partial S}{\partial u_1}^2)=1$ Jacobi's equation-Separable
- $\sqrt{\frac{u_2-u_0}{f(u_2)(u_2-\alpha)}}du_2-\sqrt{\frac{u_1-u_0}{f(u_1)(u_1-\alpha)}}du_1=0$, α constant.

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- Integrability by the method of sparation of variables- Inverse method (See Mechanics by Landau and Lifschitz).
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- Basic problem: How to write the Hamiltonian for optimal problems in which tangents of curves (i.e., controls) are constrained by state constraints?
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  $F_k = p_k^2 + \sum_{j=0,j\neq k}^{n} \frac{(x_jp_k - x_kp_j)^2}{(a_k-a_j)}), k = 0,\ldots,n,$
  $A = diag(a_0,\ldots,a_n)$.
- There are n functionally independent integrals generated by $F_0,\ldots,F_n$ ($\sum_{i=0}^{n} F_i = H = 1$), and $\{F_i, F_j\} = 0$.
- Jacobi's problem is Liouville integrable.
- Arnold's querry: What are the hidden symmetries that account for the integrability of Jacobi's problem?
- My earlier work suggested that "all" integrable systems (tops, elastic problems, Toda lattices,etc.) are the projections of left-invariant Hamiltonian systems on Lie groups with additional symmetries.
- Question: Are "hidden" symmetries hidden in Lie algebras and Lie groups?

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Main point of the talk- Spectral representation and hidden symmetries

- The elliptic geodesic problem on S^n is equivalent to Jacobi's problem on E^n.
- The cotangent bundle of the sphere S^n is a coadjoint orbit.
- The Hamiltonian system of the geodesic problem on S^n when represented on the coadjoint orbit can be seen as the restricition of a left invariant Hamiltonian system on the semidirect product $SO_{n+1}(R) \rtimes Sym_{n+1}$
- This Hamiltonian system admits a spectral matrix representation - the spectral invariants generate the integrals of motion for the elliptic geodesic problem which correspond to the integrals $F_k, k = 0, 1, \ldots, n$ on the ellipsoid.
- This recognition identifies a large class of intergrable problems in which Jacobi's problem is only a particular case.

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Background and history

Elliptic Geodesic problem on the sphere

Representation on coadjoint orbits

Integrability

Left invariant optimal control problems on Lie groups

Geometric interpretations

Open questions

Overlay

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- The " right" Hamiltonian H is given by $H = H_0 + \lambda_3 C_3 + \lambda_4 C_4$, $C_3 = ||x||^2 - 1$, $C_4 = (x, p)$.
- $\{H_0, C_3\} = \{H_0, C_4\} = 0 \Rightarrow \lambda_3 = \frac{1}{2}, \lambda_4 = 0$.
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Semidirect product

- $\mathfrak{g} = sl_{n+1}(R)$, $\mathfrak{p} = \{A \in \mathfrak{g} : A^T = A\}$, $\mathfrak{k} = so_{n+1}(R)$.
- The Killing form $\langle A, B\rangle = -\frac{1}{2}Tr(AB)$.
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- $\mathfrak{g}^*$ the dual of $\mathfrak{g}$.
- $l \in \mathfrak{g}^* \longleftrightarrow L \in \mathfrak{g}$ iff $l(X) = \langle L, X\rangle \forall X \in \mathfrak{g}$.
- Proposition. The coadjoint orbit of the semidirect product $G = \mathfrak{p} \ltimes SO_{n+1}(R)$ through a matrix $X_0 = x_0 \otimes x_0 - \frac{||x_0||^2}{n+1}I$ is equal to $\{P + K) : P = x \otimes x - \frac{||x||^2}{n+1}I, K = x \wedge p, ||x|| = ||x_0||, p \cdot x = 0\}$ and is isomorphic to the cotangent bundle of the sphere $||x|| = ||x_0||$ (T. Ratiu, 1980).
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- Geodesic equations on the coadjoint orbit (after the reparametrization $\frac{ds}{dt} = (x(t) \cdot A^{-1}x(t))^{-1}$:
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  $F_k(x, p) = x_k^2 + \sum_{j=1, j\neq k}^{n} \frac{(x_j p_k - x_k p_j)^2}{a_k - a_j}, k = 0, \ldots, n$, or
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- Spectral representation:$\frac{dL_\lambda}{ds} = [\Omega_\lambda, L_\lambda]$ where
- $L_\lambda = P - \lambda K - \lambda^2 A, \Omega_\lambda = A^{-1}KA^{-1} - \lambda A^{-1}$.
- Proof: $\frac{dL_\lambda}{ds} = [A^{-1}KA^{-1}, P] - \lambda([A^{-1}KA^{-1}, K] + [A^{-1}, P]) = [A^{-1}KA^{-1} - \lambda A^{-1}, P - \lambda K] - \lambda^2[A^{-1}, K] = [\Omega_\lambda, L_\lambda]$
- $F(z) = R_z x(t)\cdot x(t)(1 + R_z p(t)\cdot p(t)) - (R_z x(t)\cdot p(t))^2$, $R_z = (zI - A)^{-1}$, is constant along the geodesic flow.(Newmann problem, J. Moser, Chern Symposium(1979)) (The spectral invariants)
- Constants of motion
  $F_k(x,p) = x_k^2 + \sum_{j=1, j\neq k}^n \frac{(x_j p_k - x_k p_j)^2}{a_k - a_j}, k = 0, \ldots, n$, or
- $G_k(y,q) = q_k^2 + \sum_{j=1, j\neq k}^n \frac{(y_j q_k - y_k q_j)^2}{a_k - a_j}, k = 0, \ldots, n$

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- G is a semisimple Lie group, $\mathfrak{g}$ is its Lie algebra.
- Assumption. $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, $\mathfrak{p}$ is a vector space (Cartan space), $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}, [\mathfrak{p}, \mathfrak{k}] = \mathfrak{p}$.
- $B \in \mathfrak{p}$ is said to be regular if $\{X \in \mathfrak{p} : [X, B] = 0\}$ is a maximal Abelian algebra in $\mathfrak{p}$.
- Natural control problem: $\frac{dg}{dt} = g(B + u(t)), u(t) \in \mathfrak{k}$, B regular.
- Controllability: Given g_0 and g_1 both in G there exist $T > 0$ and a control $u(t) \in \mathfrak{k}$ such that the solution $g(t)$ with $g(0) = g_0$ also satisfies $g(T) = g_1$.
- Suppose now that Q is a positive definite quadratic form on $\mathfrak{k}$. Then for any boundary conditions g_0 and g_1 there exists $T > 0$ such that the optimal control problem $Min\frac{1}{2}\int_0^T Q(u(t), u(t))dt$ has a solution.

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Left invariant Hamiltonians

- Special case $\langle X, Y\rangle = -Tr(ad(X) \circ ad(Y))$ (Cartan-Killing form).
- Suppose $\phi : \mathfrak{k} \to \mathfrak{k}$ is a linear automorphism such that $\langle \phi(u), u\rangle > 0,$ and $\langle \phi(u), v\rangle = \langle u, \phi(v)\rangle \forall u, v \in \mathfrak{k}$. Then $Q(u, u) = \langle \phi(u), u\rangle$.
- Background: $T^*G = \mathfrak{g}^* \times G$. $l \in \mathfrak{g}^* \longleftrightarrow L \in \mathfrak{g}$ via the Killing form. $L = P + K, P \in \mathfrak{p}, K \in \mathfrak{k}$.
- Normal extrema $Max_u(-\frac{1}{2}\langle \phi(u), u\rangle + \langle B, P\rangle + \langle u, K\rangle$ occurs when $u = \phi^{-1}(K)$. Hence normal extrema are integral curves of a single Hamiltonian $H = \frac{1}{2}\langle K, \phi^{-1}(K)\rangle + \langle B, P\rangle$.
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- $\frac{1}{2}\int_0^T ||K(t)||^2\,dt = \frac{1}{2}\int_0^T \kappa(t)^2\,dt + const$
- Elastica When $K_0 = 0$ $\kappa(t)$ is expressible in terms of elliptic functions, $\kappa^2(t)\tau(t) = constant$ and higher curvatures of $x(t)$ are all zero. (P. Griffiths,1983, and V. Jurdjevic-F.M. Perez, 2002)

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$$H = \frac{1}{2}\langle K, K\rangle_{\mathfrak{h}} + \langle A, P\rangle_{\mathfrak{h}} = \frac{1}{2}(||p|^2||q||^2) + (Aq, q).$$

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Some natural questions

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- Integration on the quotient spaces G/K where K is the Lie group whose Lie algebra is $\mathfrak{k}$.
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