The topology of a random intersection of real quadrics

Antonio Lerario

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Real random polynomials

$$f(t) = a_0 + a_1 t + \ldots + a_d t^d$$

or equivalently

$$f(x_0, x_1) = a_0 x_0^d + a_1 x_0^{d-1} x_1 + \ldots + a_d x_1^d$$

The coefficients of f are gaussian random variables, i.e.:

$$\mathbb{P}\{a_i \leq c\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\frac{x^2}{2\sigma_i^2}} dx, \quad (a_i \sim N(0, \sigma_i))$$

Question

What is the expected value E_d of the number of real roots of f?

The average number of real roots of a random polynomial

- $a_i \sim N(0,1)$, $i = 0, \ldots d$, independent:

$$\lim_{d\to\infty} \frac{E_d}{\log d} = \frac{2}{\pi}$$

  (Kac, '43)

- $a_i \sim N(0, \binom{d}{i})$, $i = 0, \ldots d$, independent:

Theorem (Edelman and Kostlan, '95)

$$E_d = \sqrt{d}$$

What is the meaning of this distribution?

Gaussian distributions

Gaussian distribution on a vector space $V \iff$ scalar product on V

$$\mathbb{P}\{v \in A\} = \frac{1}{c} \int_A e^{-\frac{\langle v, v \rangle}{2}} dv$$

$V = H_{d,1} = \{f \text{ homogeneous of degree } d \text{ in two variables}\}$

$$\langle f, g \rangle = \int_{\mathbb{C}^2} f(z) \overline{g(z)} e^{-\|z\|^2} dz$$

this gives the previous distribution, i.e. $a_i \sim N(0, \binom{d}{i})$.

Weyl distribution

$$H_{d,n} = \{f \text{ real, homogeneous of degree } d \text{ in } n+1 \text{ variables}\}$$

$$\langle f, g\rangle = \int_{\mathbb{C}^{n+1}} f(z)\overline{g(z)} e^{-\|z\|^2} dz$$

Definition (Weyl distribution)

$$f(x) = \sum_{\alpha} f_\alpha x^\alpha, \quad \alpha = (\alpha_0, \ldots, \alpha_n)$$

$$f_\alpha \sim N(0, \frac{d!}{\alpha_0! \cdots \alpha_d!})$$

Real zeroes vs Complex zeroes

$$f \in H_{d,n} : \quad Z_{\mathbb{R}}(f) \subset \mathbb{R}\mathrm{P}^n, \quad Z_{\mathbb{C}}(f) \subset \mathbb{C}\mathrm{P}^n$$

Idea

We wish to compare $Z_{\mathbb{R}}(f)$ and $Z_{\mathbb{C}}(f)$.

$$f \in H_{d,1} : \quad \sharp Z_{\mathbb{R}}(f) \leq \sharp Z_{\mathbb{C}}(f)$$

$$f \text{ Weyl distributed} : E_d = \sqrt{d} \leq d.$$

The expected volume

More generally let

$$X_{\mathbb{R}} = Z_{\mathbb{R}}(f_1, \dots, f_k) \subset \mathbb{R}\mathrm{P}^n$$

and $\mathbb{R}\mathrm{P}^n$ with Fubiny-Study density induced from $\mathbb{C}\mathrm{P}^n$:

$$\mathrm{Vol}(X_{\mathbb{R}}) \leq \mathrm{Vol}(X_{\mathbb{C}}) = d_1 \cdots d_k$$

Theorem

If $f_1, \dots, f_k$ independent and Weyl distributed:

$$\mathbb{E}\, Vol(X_{\mathbb{R}}) = \sqrt{d_1, \dots, d_k}\, Vol(\mathbb{R}\mathrm{P}^{n-k})$$

(Shub and Smale '00, Burgisser '07)

The curvature polynomial

$M \subset \mathbb{RP}^n$ of dimension m

$$T(M,\epsilon) = \{y \in \mathbb{RP}^n \mid d(y,M) \leq \epsilon\}$$

For $\epsilon > 0$ small enough (Weyl):

$$\mathrm{Vol}(T(M,\epsilon)) = \sum_{0 \leq e \leq m,\, e \,\mathrm{even}} K_{s+e}(M) J_{n,s+e}(\epsilon)/2$$

where $J_{n,k}(\epsilon) = \int_0^\epsilon (\sin t)^{k-1} (\cos t)^{n-k} dt$ and K_{s+e} depend only on the intrinsic geometry of M (curvature coefficients).

$$\mu(M,x) = \sum_{0 \leq e \leq m,\, e \,\mathrm{even}} \frac{K_{s+e}(M)}{\mathrm{Vol}(S^{m-e})\mathrm{Vol}(S^{s+e-1})} x^e$$

$\mu(M,0)$ is the normalized volume and $\mu(M,1) = \chi(M)$ (m even). Burgisser has computed the expected curvature polynomial!

Number of points = zero dimensional volume

$$\mathbb{E}\mathrm{Vol}(X_{\mathbb{R}}) = \sqrt{d_1, \ldots, d_k}\,\mathrm{Vol}(\mathbb{R}\mathrm{P}^{n-k})$$

For $k = n = 1$ this is

$$E_d = \sqrt{d}$$

Idea

Number of points = zero dimensional volume

$$\sharp\{\text{real roots of } f\} \leq \sharp\{\text{complex roots of } f\}$$

generalizes as

$$\mathrm{Vol}(Z_{\mathbb{R}}(f)) \leq \mathrm{Vol}(Z_{\mathbb{C}}(f)).$$

Number of points = total Betti number

$$b(X) = \sum b_i(X), \quad b_i(X) = \mathrm{rk} H_i(X)$$

$\tilde{b}_i(X)$ is the number of $i+1$ dimensional *holes* in X.

Idea

Number of points in $Z_{\mathbb{R}}(f) = b(Z_{\mathbb{R}}(f))$.

$$\sharp\{\text{real roots of } f\} \leq \sharp\{\text{complex roots of } f\}$$

generalizes as

$$b(X_{\mathbb{R}}) \leq b(X_{\mathbb{C}}) \quad (Smith's inequality)$$

Smith's inequality $b(X_{\mathbb{R}}) \leq b(X_{\mathbb{C}})$ for curves

Example: $X_{\mathbb{R}}$ real curve in $\mathbb{R}\mathrm{P}^2$ of degree d.

$$\text{Harnack: } X_{\mathbb{R}} \text{ has at most } \frac{(d-1)(d-2)}{2} + 1 \text{ ovals.}$$

$$\text{Genus formula: } X_{\mathbb{C}} \text{ is a Riemann surface with } g = \frac{(d-1)(d-2)}{2}$$

$$b(X_{\mathbb{R}}) = 2b_0(X_{\mathbb{R}}) \leq 2g + 2 = b(X_{\mathbb{C}})$$

Curves with $g+1$ ovals are called maximal and are *extremely* hard to built starting from their coefficients.

Average number of ovals of a real curve

The answer is not known.

Curves with approximatively d^2 components exponentially rarefact (Gayet and Welschinger, '11).

$$\lim_{d\to\infty} \frac{\mathbb{E}[b(X_{\mathbb{R}})]}{d} \leq \pi \qquad \text{(Sarnak, '11)}$$

Intersection of real quadrics

From now on

$$X_{\mathbb{R}} = Z_{\mathbb{R}}(q_1, \ldots, q_k), \quad q_i \in H_{2,n}$$

Idea

To compare $b(X_{\mathbb{R}})$ and $b(X_{\mathbb{C}})$.

$b(X_{\mathbb{C}})$ is known (for regular intersections):

- $k = 1 :$ $\quad b(X_{\mathbb{C}}) = n + \frac{1}{2}(1 + (-1)^{n+1})$
- $k = 2 :$ $\quad b(X_{\mathbb{C}}) = 2n - \frac{1}{2}(1 + (-1)^{n+1})$
- $k = 3 :$ $\quad b(X_{\mathbb{C}}) = n^2 + \frac{1}{2}(5 + 3(-1)^{n}) = n^2 + O(n)$

What about $X_{\mathbb{R}}$?

Topology of intersection of real quadrics

$$W = \mathrm{span}\{q_1, \ldots, q_k\}$$

$$\Sigma = \{q \mid \ker(q) \neq 0\} \cap \{\|q\|^2 = 1\}$$

$$\Sigma_W = \Sigma \cap W$$

Theorem (Agrachev '90, Agrachev and L. '11)

$$b(X_{\mathbb{R}}) \approx b(\Sigma_W)$$

- $k = 1:$ $\qquad b(X_{\mathbb{R}}) = 2 \min \mathrm{i}^+|_W$
- $k = 2:$ $\qquad b(X_{\mathbb{R}}) \approx 2 \min \mathrm{i}^+|_W + n + \frac{1}{2} b(\Sigma_W)$
- $k = 3:$ $\qquad b(X_{\mathbb{R}}) \approx 2 \min \mathrm{i}^+|_W + n + \frac{1}{2} b(\Sigma_W)$

Complexity of intersection of real quadrics

$$X_{\mathbb{R}} = Z_{\mathbb{R}}(q_1, \ldots, q_k)$$

$\Sigma_W \subset S^{k-1}$ hypersurface of degree $n+1$ (number of variables)

- $k = 1 :$ $\Sigma_W = \emptyset$
- $k = 2 :$ $\Sigma_W =$ points on S^1
- $k = 3 :$ $\Sigma_W =$ curve of degree $n+1$ on S^2

- $k = 1 :$ $b(\Sigma_W) = 0$
- $k = 2 :$ $b(\Sigma_W) \leq 2n + 2$
- $k = 3 :$ $b(\Sigma_W) \leq n^2 + O(n)$

Theorem (Barvinok 99', Agrachev and L. '11)

$$b(X_{\mathbb{R}}) \leq n^{O(k-1)}$$

The topology of a random quadratic hypersurface in $\mathbb{RP}^n$

q real quadratic form in $n+1$ variables.

$$q(x) = \langle x, Qx \rangle, \quad Q \in \mathrm{Sym}_{n+1}(\mathbb{R})$$

Idea

q Weyl distributed is equivalent to $Q \in GOE$.

$$b(Z_{\mathbb{R}}(q)) = 2 \min \mathrm{i}^+|_W$$

$W = \mathrm{span}(q)$ hence $\min \mathrm{i}^+|_W = \min\{\mathrm{i}^+(q), \mathrm{i}^+(-q)\}$

Question

$$\mathbb{E} \min\{\mathrm{i}^+(q), \mathrm{i}^+(-q)\} = ?$$

Wigner's semicircular law

$$\text{ESD}: \quad \mu_n = \frac{1}{n}\sum_{i=0}^{n} \delta_{\lambda_i(Q)/\sqrt{n}} \quad Q \in GOE$$

$$\text{SC}: \mu_{sc} = \frac{1}{2\pi}(4-x^2)_+^{1/2} dx$$

Theorem (Wigner)

For every $\psi \in C_c^0(\mathbb{R})$:

$$\lim_{n\to\infty} \mathbb{E} \int_{\mathbb{R}} \psi d\mu_n = \int_{\mathbb{R}} \psi d\mu_{sc}$$

For every $A \subset \mathbb{R}$ the expected number of eigenvalues of $Q/\sqrt{n}$ in A divided by n is asymptotically as $\int_A d\mu_s c$

The topology of a random quadratic hypersurface in $\mathbb{RP}^n$

$\mathrm{i}^+(q) =$ number of eigenvalues in $[0, \infty)$

$$\lim_{n\to\infty} \frac{\mathbb{E}\min\{\mathrm{i}^+(q), \mathrm{i}^+(-q)\}}{n} = \frac{1}{2}$$

Theorem (L. '12)

$$\lim_{n\to\infty} \frac{\mathbb{E}[b(X_\mathbb{R})]}{n} = 1$$

Since $b(X_\mathbb{C}) = n + \frac{1}{2}(1 + (-1)^{n+1})$, then:

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{b(X_\mathbb{R})}{b(X_\mathbb{C})}\right] = 1$$

Thus Smith's inequality $b(X_\mathbb{R}) \leq b(X_\mathbb{C})$ is expected to be sharp as we let n growth.

The random intersection of two quadrics in $\mathbb{RP}^n$

$$X_{\mathbb{R}} = Z_{\mathbb{R}}(q_1, q_2)$$

$$b(X_{\mathbb{R}}) = 2\min \mathrm{i}^+|_W + n + \frac{1}{2} b(\Sigma_W) + \epsilon, \quad \epsilon \in \{0, 1, 2\}$$

- Wigner's semicircular law:

$$\lim_{n\to\infty} \frac{\mathbb{E}\min \mathrm{i}^+|_W}{n} = \frac{1}{2}$$

- What about

$$\lim_{n\to\infty} \frac{\mathbb{E} b(\Sigma_W)}{n} = ?$$

Gap probability in GOE

Integral geometry formula gives:

$$\mathbb{E}b(\Sigma_W) = \frac{\mathrm{Vol}(\Sigma)}{\mathrm{Vol}(S^{N-2})}$$

Eckart-Young: $d_F(q, \Sigma) = \sigma(Q)^{-1}$

$$\mathrm{Vol}(\Sigma) = \lim_{\epsilon \to 0} \frac{1 - \mathbb{P}\{\text{no eigenvalues in } (-\epsilon, \epsilon)\}}{2\epsilon}$$

$$\mathrm{Vol}(\Sigma) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} \mathrm{Vol}(S^{\frac{n(n+1)}{2}-1})$$

$$\lim_{n \to \infty} \mathbb{E}b(\Sigma_W) = \lim_{n \to \infty} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} \frac{\mathrm{Vol}(S^{N-1})}{\mathrm{Vol}(S^{N-2})} = 0$$

The random intersection of two quadrics in $\mathbb{RP}^n$

$$X_{\mathbb{R}} = Z_{\mathbb{R}}(q_1, q_2)$$

$$b(X_{\mathbb{R}}) = 2 \min \mathrm{i}^+|_W + n + \frac{1}{2} b(\Sigma_W) + \epsilon, \quad \epsilon \in \{0, 1, 2\}$$

- $\lim_{n\to\infty} \frac{\mathbb{E} \min \mathrm{i}^+|_W}{n} = \frac{1}{2}$
- $\lim_{n\to\infty} \frac{\mathbb{E} b(\Sigma_W)}{n} = 0$

Theorem (L. '12)

Let $X_{\mathbb{R}}$ be the intersection of two real random quadrics in $\mathbb{RP}^n$, then:

$$\lim_{n\to\infty} \frac{\mathbb{E} b(X_{\mathbb{R}})}{2n} = 1$$

Smith's inequality

Since $b(X_{\mathbb{C}}) = 2n - (1 + (-1)^{n+1})$, then:

$$\lim_{n \to \infty} \mathbb{E}\left[\frac{b(X_{\mathbb{R}})}{b(X_{\mathbb{C}})}\right] = 1$$

Thus also in this case Smith's inequality $b(X_{\mathbb{R}}) \leq b(X_{\mathbb{C}})$ is expected to be sharp as we let n growth.

Open Question

What happens with more quadrics?

Random Hilbert's 16th problem

$$X_{\mathbb{R}} = Z_{\mathbb{R}}(q_1, q_2, q_3)$$

$$b(X_{\mathbb{R}}) \approx b(\Sigma_W)$$

Σ_W is the curve on S^2 given by $\det(x_1 Q_1 + x_2 Q_2 + x_3 Q_3) = 0$ this curve has degree $n+1$, hence:

$$b(\Sigma_W) \leq n^2 + O(n)$$

$$b(X_{\mathbb{C}}) = n^2 + O(n)$$

$$\lim_{n\to\infty} \mathbb{E}\frac{b(\Sigma_W)}{n^2} = \lim_{n\to\infty} \mathbb{E}\frac{b(X_{\mathbb{R}})}{n^2} = \lim_{n\to\infty} \mathbb{E}\frac{b(X_{\mathbb{R}})}{b(X_{\mathbb{C}})} = 1$$

Thus if Smith's inequality is expected to be sharp for $X_{\mathbb{R}}$, we can produce almost maximal curves with high probability!

Random Hilbert's 16th problem

Consider the space W_d of $f : S^2 \to \mathbb{R}$ spherical harmonic of degree d with the gaussian density given by:

$$\langle f, g \rangle = \int_{S^2} fg\omega_{S^2}$$

Theorem (Nazarov and Sodin, 07')

There exists $c_1 > 0$ such that for a random spherical harmonic of degree d:

$$\lim_{d\to\infty} \frac{\mathbb{E}b(Z(f))}{d^2} = c_1$$

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