Control and mixing for 2D Navier–Stokes equations with space-time localised force

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Outline

1. Introduction: control problem and its stochastic counterpart
2. Two-dimensional Navier–Stokes system
   - Initial-boundary value problem
   - Problem of ergodicity
3. Main result and some ideas of its proof
   - Formulation
   - “Right” controllability properties
   - Proof of the squeezing property (with V. Barbu & S. Rodrigues)

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Abstract setting

Let H be a Hilbert space let $E \subset H$ be a closed subspace. Consider the equaton

$$\partial_t u = F(u) + \eta(t), \quad u(t) \in H, \quad \eta(t) \in E, \tag{1}$$

supplemented with the initial condition

$$u(0) = u_0 \in H. \tag{2}$$

We assume that problem (1), (2) is well posed and denote by $S_t(u_0, \eta)$ its resolving operator.

Control problem. Assuming that η is a control with range in E, investigate various controllability properties, such as exact or approximate controllability, stabilisation of a given solution, etc.

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Random dynamical system. Let λ be a probability measure on $L^2(0,1;E)$ and let $\{\eta_k\}$ be a sequence of i.i.d. random variables in $L^2(0,1;E)$ such that $\mathcal{D}(\eta_k)=\lambda$, where $\mathcal{D}(\xi)$ denotes the law of a random variable ξ.

Define the mapping

$$\varphi_k : u_0 \mapsto u(k), \quad k \geq 0,$$

where $u(t)=S_t(u_0,\eta)$ is the solution of (1), (2) with η such that

$$\eta(t)=\eta_k(t-k+1) \quad \text{for } t\in(k-1,k), \quad k\geq 1.$$

Then $\{\varphi_k : H\to H, k\geq 0\}$ form an RDS in H, and the problem is to study the large-time asymptotics of its trajectories.

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General principle

We shall use the example of 2D Navier–Stokes equations to justify the following general principle:

"Right" controllability properties imply ergodicity

Ergodicity means the existence of a probability measure μ on H such that

$$\mathcal{D}(\varphi_k(u_0)) \rightharpoonup \mu \quad \text{as } k \to \infty. \tag{3}$$

"Right" controllability properties are the following ones:

- **Approximate controllability in infinite time;**
- **Squeezing:** *for any $v, v' \in H$ there is a finite-dimensional map $\Psi_{v,v'}$ acting in $L^2(0,1;E)$ such that*

$$\left\| S_1(v,\zeta) - S_1(v',\zeta + \Psi_{v,v'}(\zeta)) \right\| \le \frac{1}{2} \|v - v'\|. \tag{4}$$

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Equations and phase space

Let $D \subset \mathbb{R}^2$ be a domain with a smooth boundary ∂D. Consider the problem

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \quad x \in D, \tag{5}$$

$$u(0,x) = u_0(x), \quad u\big|_{\partial D} = 0. \tag{6}$$

Here $u = (u_1, u_2)$ and p are unknown velocity field and pressure, $\nu > 0$ is the viscosity, f is an external force, and

$$\langle u, \nabla \rangle = u_1(t,x)\partial_1 + u_2(t,x)\partial_2.$$

Problem (5), (6) is well posed in the space

$$H = \{u \in L^2(D, \mathbb{R}^2) : \operatorname{div} u = 0 \text{ in } D, \langle u, \boldsymbol{n} \rangle = 0 \text{ on } \partial D\},$$

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Random force

Let $Q \subset (0,1) \times D$ be an open subset and let $\{\eta_k\}$ be a sequence of i.i.d. random variables in the space $H_0^1(Q, \mathbb{R}^2)$ regarded as a subspace in $L^2((0,1) \times D, \mathbb{R}^2)$. Assume that

$$f = h + \eta, \quad h \in H, \quad \eta = \sum_{k=1}^{\infty} I_{(k-1,k)}(t)\, \eta_k(t-k+1, x),$$

and define $\varphi_k : H \to H$ by the relation

$$\varphi_k(u_0) = u(k, \cdot), \quad u(t,x) = S_t(u_0, \eta) \text{ is the solution of (5), (6).}$$

The independence of η_k ensures that the trajectories of the RDS φ_k are Markov processes in H, while the k-independence of their law implies that those Markov processes are homogeneous in time.

Metric in the space of measures and ergodicity

Define the Kantorovich–Wasserstein distance (metrising the weak convergence) on the space $\mathcal{P}(H)$ of probabilities on H:

$$\|\mu_1 - \mu_2\|_L^* = \sup_{\|g\|_L \le 1} \left| \int_H g \, d\mu_1 - \int_H g \, d\mu_2 \right|, \quad \mu_1, \mu_2 \in \mathcal{P}(H),$$

where for a function $g : H \to \mathbb{R}$ we set

$$\|g\|_L = \sup_{u \in H} |g(u)| + \sup_{0 < \|u-v\|_H \le 1} \frac{|g(u) - g(v)|}{\|u - v\|_H}.$$

Problem of ergodicity. Find a measure $\mu \in \mathcal{P}(H)$ such that

$$\|\mathcal{D}(\varphi_k(u_0)) - \mu\|_L^* \le C(1 + \|u_0\|_H) e^{-\gamma k}, \quad k \ge 0, \qquad (7)$$

where C and γ are positive constants not depending on u_0.

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Formulation of the main result

We first specify the structure of random variables η_k. Let $\chi \in C_0^\infty(Q)$ be a nonzero function and let $\{\varphi_j\} \subset H^1(Q, \mathbb{R}^2)$ be a orthonormal basis in $L^2(Q, \mathbb{R}^2)$. We assume that

$$\eta_k(t,x) = \sum_{j=1}^{\infty} b_j \xi_{jk} \psi_j(t,x), \quad \psi_j = \chi \varphi_j, \tag{8}$$

where $b_j \in \mathbb{R}$ go to zero sufficiently fast and ξ_{jk} are independent random variables such that

$$\mathcal{D}(\xi_{jk}) = \rho_j(r)dr, \quad \rho_j \in C_0^1([-1,1]).$$

Main Theorem

Let $h \in H^1(D, \mathbb{R}^2)$. Then there are $\varepsilon > 0$ and $N \geq 1$ depending on $\nu > 0$ such that the RDS $\{\varphi_k\}$ is ergodic, provided that

$$\|h\|_H \leq \varepsilon, \qquad b_j \neq 0 \quad \text{for} \quad 1 \leq j \leq N. \tag{9}$$

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Sufficient condition for ergodicity

The existence and uniqueness of a measure $\mu \in \mathcal{P}(H)$ and its stability (without the rate of convergence) are consequences of the following two properties in which $\hat{u} \in H$ is a given function:

- **Recurrence.** For any $d > 0$ and $u_0 \in H$, the random time
$$\tau(u_0) = \min\{k \geq 0 : \|\varphi_k(u_0) - \hat{u}\|_H \leq d\}$$
is almost surely finite.
- **Stability.** There is a continuous function $\varkappa(d)$ defined for $d \in [0, 1]$ and going to zero with d such that
$$\sup_{\|u_0-\hat{u}\|_H \leq d} \left\|\mathcal{D}(\varphi_k(u_0)) - \mathcal{D}(\varphi_k(\hat{u}))\right\|_L^* \leq \varkappa(d). \tag{10}$$

The first property follows from the *approximate controllability to $\hat{u}$ in infinite time*, while the second can be established by combining the squeezing property with a general result on the transformation of measures by smooth mappings.

Approximate controllability in infinite time

Let us denote by $\mathcal{K}$ the support of the law for η_k. Thus, $\mathcal{K}$ is a compact subspace in $H_0^1(Q, \mathbb{R}^2)$. Suppose there is $\hat{u} \in H$ for which the following properties hold.

Approximate controllability. *For any $R, d > 0$ there is an integer $l \geq 1$ such that, given $u_0 \in B_H(R)$, we can find controls $\zeta_1, \dots, \zeta_l \in \mathcal{K}$ satisfying the inequality*

$$\|S_l(u_0; \zeta_1, \dots, \zeta_l) - \hat{u}\|_H \leq d, \tag{11}$$

where $S_l(u_0; \zeta_1, \dots, \zeta_k)$ stands for the restriction at $t = l$ of the solution for the Navier–Stokes system with the right-hand side

$$f(t, x) = h(x) + \sum_{k=1}^{l} I_{(k-1,k)}(t)\, \zeta_k(t - k + 1, x).$$

This property is well understood due to the works of *Coron* (1995-1996), *Fursikov–Imanuvilov* (1996-2000), and others. It is trivially satisfied when $\|h\| \ll 1$.

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Squeezing

Let $\mathcal{E}_m \subset H_0^1(Q, \mathbb{R}^2)$ be the vector span of $\psi_j(t,x)$, $j = 1, \dots, m$, and $B_X(R)$ be the closed ball in X of radius R centred at zero.

Squeezing. $\forall\, R > 0$ there is $d_0 > 0$, $m \geq 1$, and a mapping

$$\Phi : B_H(R) \times B_{H_0^1(Q,\mathbb{R}^2)}(R) \to \mathcal{L}(H, \mathcal{E}_m)$$

(where $\mathcal{L}(H, \mathcal{E}_m)$ is the space of continuous linear operators from H to $\mathcal{E}_m$) with the following properties:

- **Contraction.** For any $u_0, u_0' \in B_H(R)$ with $\|u_0 - u_0'\| \leq d_0$ and any $\zeta \in B_{H_0^1(Q,\mathbb{R}^2)}(R)$ we have

$$\left\| S_1(u_0, \zeta) - S_1\left(u_0', \zeta + \Phi(u_0, \zeta)(u_0' - u_0)\right)\right\| \leq \frac{1}{2}\|u_0 - u_0'\|. \tag{12}$$

- **Regularity.** The mapping $\Phi(u_0, \zeta)$ is regular and uniformly Lipschitz with respect to its arguments.

Reduction to the linear case

We seek the perturbed solution in the form $u' = u + v$. Then v must satisfy the equations

$$\partial_t v + \langle v, \nabla \rangle v + \langle u, \nabla \rangle v + \langle v, \nabla \rangle u - \nu \Delta v + \nabla p = \chi(\mathrm{P}_m \xi),$$
$$\mathrm{div}\, v = 0, \quad v(0) = v_0 := u_0' - u_0, \quad v\big|_{\partial D} = 0,$$

where $\xi \in L^2((0,1) \times D, \mathbb{R}^2)$ is an unknown control and P_m is the orthogonal projection to $\mathrm{span}\{\varphi_1, \dots, \varphi_m\}$. We wish to find ξ such that

$$\|v(1; v_0, \xi)\| \le \frac{1}{2} \|v_0\|. \tag{13}$$

Since $\|v_0\| \le d_0 \ll 1$, a standard perturbation argument shows that it suffices to consider the linearised equation around zero:

$$\partial_t v + \langle u, \nabla \rangle v + \langle v, \nabla \rangle u - \nu \Delta v + \nabla p = \chi(\mathrm{P}_m \xi). \tag{14}$$

Foiaş–Prodi property

Let $\{e_j\}$ be an ONB in H consisting of the eigenfunctions of the Stokes operator. Suppose for any $N \geq 1$ and $\varepsilon > 0$ we found an integer $m \geq 1$ and a control $\xi = \xi(v_0, u) \in \mathcal{E}_m$ such that

$$\|\Pi_N v(1; v_0, \xi)\| \leq C\,\varepsilon\,\|v_0\|, \quad \|\xi\|_{L^2(Q,\mathbb{R}^2)} \leq C\,\|v_0\|, \tag{15}$$

where Π_N is the orthogonal projection in H onto the vector span of $e_1, \dots, e_N$ and $C > 0$ is independent of N and ε. Then

$$
\begin{aligned}
\|v(1; v_0, \xi)\| &\leq \|\Pi_N v(1)\| + \|(Id - \Pi_N)v(1)\| \\
\text{(15) + Poincaré} &\leq C\,\varepsilon\,\|v_0\| + \delta_N\,\|v(1)\|_{H^1} \\
\text{continuity of } v \text{ in } (v_0, \xi) &\leq C\,\varepsilon\,\|v_0\| + C_1\delta_N\big(\|\xi\|_{L^2(Q,\mathbb{R}^2)} + \|v_0\|\big) \\
\text{(15)} &\leq \big(C\,\varepsilon + C_1(C+1)\delta_N\big)\|v_0\| \\
\varepsilon \ll 1, N \gg 1 &\leq \frac{1}{4}\|v_0\|.
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Minimisation problem

Set $B(u)v = \langle u, \nabla\rangle v + \langle v, \nabla\rangle u$ and consider the following linear-quadratic minimisation problem:

$$J_\varepsilon(v,\xi) = \frac{1}{2}\int_0^1 \|\xi(t)\|^2 dt + \frac{1}{\varepsilon}\|\Pi_N v(1)\|^2 \to \min,$$

$$\dot v + B(u)v - \nu\Delta v + \nabla p = \chi(\mathsf{P}_m\xi), \quad \operatorname{div} v = 0,$$

$$v\big|_{\partial D} = 0, \quad v(0) = v_0.$$

This problem has a unique solution (v,ξ) for any ε, m, and N. The optimality condition results in

$$\frac{2}{\varepsilon}\|\Pi_N v(1)\|^2 + \int_0^1 \|\mathsf{P}_m(\chi g)\|^2 dt = -(g(0), v_0)_{L^2}, \tag{16}$$

where g is the solution of the dual problem:

$$-\dot g + B^*(u)g - \nu\Delta g + \nabla\pi = 0, \quad \operatorname{div} g = 0, \quad g(1) = -\frac{2}{\varepsilon}\Pi_N v(1). \tag{17}$$

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Truncated observability inequality

Proposition

There is $C > 0$ such that for any $N \geq 1$ and a sufficiently large $m = m(N) \geq 1$ any solution of (17) satisfies the inequality

$$\|g(0)\|^2 \leq C \int_0^1 \|\mathrm{P}_m(\chi g)\|^2 dt. \tag{18}$$

Combining (18) with (16), we get

$$\frac{1}{\varepsilon}\|\Pi_N v(1)\|^2 + \int_0^1 \|\mathrm{P}_m(\chi g)\|^2 dt \leq C_2 \|v_0\|^2.$$

On the other hand, the optimality condition implies $\xi = \mathrm{P}_m(\chi g)$, whence it follows that the mapping $\Phi(u_0, \zeta) v_0 = \chi \mathrm{P}_m \xi$ satisfies the required properties.

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