Optimal observation for wave equations

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A. Agrachev’s 60th birthday, Cortona, 2012
Observation of wave equations

- $\Omega \subset \mathbb{R}^d$
- $T > 0$ fixed
- $\omega \subset \Omega$ subset of positive measure

Wave equation with Dirichlet boundary conditions

$$y_{tt} - \Delta y = 0, \quad (t, x) \in (0, T) \times \Omega,$$
$$y(t, \cdot)|_{\partial \Omega} = 0,$$
$$y(0, \cdot) = y^0 \in L^2(\Omega), \quad y_t(0, \cdot) = y^1 \in H^{-1}(\Omega)$$

$\forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad \exists! y \in C^0(0, T; L^2(\Omega)) \times C^1(0, T; H^{-1}(\Omega))$

Observable

$$z = \chi_\omega y$$
Observability

**Observability inequality**

The system is said observable (in time $T$) if there exists $C_T(\omega) > 0$ such that

$$\forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad C_T(\omega) \| (y^0, y^1) \|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_\omega y(t, x)^2 \, dx \, dt.$$ 

Bardos-Lebeau-Rauch (1992) : in the class of $C^\infty$ domains, the observability inequality holds if and only if the pair $(\omega, T)$ satisfies the Geometric Control Condition (GCC) in $\Omega$:

*Every ray of geometrical optics that propagates in $\Omega$ and is reflected on its boundary $\partial \Omega$ intersects $\omega$ in time less than $T$.***
Observability

Observability inequality

The system is said observable (in time $T$) if there exists $C_T(\omega) > 0$ such that

$$\forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad C_T(\omega)\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_\omega y(t, x)^2 \, dx \, dt.$$ 

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*Every ray of geometrical optics that propagates in $\Omega$ and is reflected on its boundary $\partial \Omega$ intersects $\omega$ in time less than $T$.***

Question

What is the "best possible" control domain $\omega$ of fixed given measure?
More precisely, two questions arise.

Let $L \in (0, 1)$ fixed.

**First problem**

Let $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ fixed. Maximize

$$G_T(\chi_\omega) = \int_0^T \int_\Omega \chi_\omega(x) y(t, x)^2 \, dx \, dt$$

over all possible subsets $\omega \subset \Omega$ of Lebesgue measure $|\omega| = L|\Omega|$.

(in what follows, denote $\mathcal{U}_L = \{\chi_\omega \mid \omega \subset \Omega, |\omega| = L|\Omega|\}$)

In this maximization problem, the optimal set $\omega$, whenever it exists, depends on the initial data $(y^0, y^1)$.

The aim of the following second problem is to discard this dependence.
Two problems

More precisely, two questions arise.

Let \( L \in (0, 1) \) fixed.

Second problem

Maximize

\[
C_T(\omega) = \inf \left\{ \frac{G_T(\chi_\omega)}{\| (y^0, y^1) \|^2_{L^2 \times H^{-1}}} \mid (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \right\}
\]

or

\[
\lim_{T \to +\infty} \frac{C_T(\omega)}{T}
\]

over all possible subsets \( \omega \subset \Omega \) of Lebesgue measure \( |\omega| = L|\Omega| \).

Remark 1

In the first case, the optimal set \( \omega \), whenever it exists, depends on \( T \), whereas it does not depend on \( T \) in the second case.
Related problems

1) What is the "best domain" for achieving HUM optimal control?

\[ y_{tt} - \Delta y = \chi_\omega u \]

2) What is the "best domain" domain for stabilization (with localized damping)?

\[ y_{tt} - \Delta y = -k\chi_\omega y_t \]

See works by
- P. Hébrard, A. Henrot : theoretical and numerical results in 1D for optimal stabilization (for all initial data).
- S. Cox, P. Freitas, F. Fahroo, K. Ito, ... : variational formulations and numerics.
- M.I. Frecker, C.S. Kubrusly, H. Malebranche, S. Kumar, J.H. Seinfeld, ... : numerical investigations (among a finite number of possible initial data).
- K. Morris, S.L. Padula, O. Sigmund, M. Van de Wal, ... : numerical investigations for actuator placements (predefined set of possible candidates), Riccati approaches.
- ...

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Optimal observation for wave equations
Spectral expression of $G_T(\chi_\omega)$

$\lambda_j, \phi_j, j \in \mathbb{N}^*$ : eigenelements

Every solution can be expanded as

$$y(t, x) = \sum_{j=1}^{+\infty} (a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)) \phi_j(x)$$

with

$$a_j = \int_\Omega y^0(x) \phi_j(x) \, dx, \quad b_j = \frac{1}{\lambda_j} \int_\Omega y^1(x) \phi_j(x) \, dx,$$

for every $j \in \mathbb{N}^*$. Moreover, $\|(y^0, y^1)\|_{L^2 \times H^{-1}} = \sum_{j=1}^{+\infty} (a_j^2 + b_j^2)$.

Then:

$$G_T(\chi_\omega) = \int_0^T \int_\omega \left( \sum_{j=1}^{+\infty} (a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)) \phi_j(x) \right)^2 \, dx \, dt = \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_\omega \phi_i(x) \phi_j(x) \, dx$$

where

$$\alpha_{ij} = \int_0^T (a_i \cos(\lambda_i t) + b_i \sin(\lambda_i t))(a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)) \, dt.$$

The coefficients $\alpha_{ij}$ depend only on the initial data $(y^0, y^1)$. 

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Conclusion:

$$G_T(\chi_\omega) = \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_{\omega} \phi_i(x) \phi_j(x) \, dx$$

with

$$\alpha_{ij} = \begin{cases} 
    a_i a_j \left( \frac{\sin(\lambda_i + \lambda_j) T}{2(i+j)} + \frac{\sin(\lambda_i - \lambda_j) T}{2(\lambda_i - \lambda_j)} \right) + a_i b_j \left( \frac{1 - \cos(\lambda_i + \lambda_j) T}{2(\lambda_i + \lambda_j)} - \frac{1 - \cos(\lambda_i - \lambda_j) T}{2(\lambda_i - \lambda_j)} \right) \\
    + a_j b_i \left( \frac{1 - \cos(\lambda_i + \lambda_j) T}{2(\lambda_i + \lambda_j)} + \frac{1 - \cos(\lambda_i - \lambda_j) T}{2(\lambda_i - \lambda_j)} \right) + b_i b_j \left( -\frac{\sin(\lambda_i + \lambda_j) T}{2(\lambda_i + \lambda_j)} + \frac{\sin(\lambda_i - \lambda_j) T}{2(\lambda_i - \lambda_j)} \right) 
    \quad \text{if } \lambda_i \neq \lambda_j, \\
    a_j^2 \left( \frac{T}{2} + \frac{\sin 2\lambda_j T}{4\lambda_j} \right) + a_j b_j \left( \frac{1 - \cos 2\lambda_j T}{2\lambda_j} \right) + b_j^2 \left( \frac{T}{2} - \frac{\sin 2\lambda_j T}{4\lambda_j} \right) 
    \quad \text{if } \lambda_i = \lambda_j. 
\end{cases}$$

The coefficients $\alpha_{ij}$ depend only on the initial data $(y^0, y^1)$. 
Solving of the first problem

Let \((y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)\) be fixed initial data, and let \(\alpha_{ij}\) be their associated coefficients defined as previously. For every \(x \in \Omega\), define

\[
\varphi(x) = \sum_{i,j=1}^{+\infty} \alpha_{ij} \phi_i(x) \phi_j(x).
\]  

(1)

Easily: \(\varphi\) is integrable on \(\Omega\). Moreover, \(G_T(\chi_\omega) = \int_\omega \varphi(x) \, dx\) for every measurable subset \(\omega\) of \(\Omega\).

First problem

\[
\sup_{\omega \subset \Omega \atop |\omega| = L|\Omega|} \int_\omega \varphi(x) \, dx
\]

Hence, clearly:

- There exists at least one optimal measurable subset \(\omega \subset \Omega\) of measure \(L|\Omega|\).
- Characterization: there exists \(\lambda \in \mathbb{R}\) such that every optimal set \(\omega\) is contained in the level set \(\{\varphi \geq \lambda\}\).
Theorem

If $\exists M, \delta > 0$ such that

$$\forall i, j \in \mathbb{N}^* \quad |\alpha_{ij}| \leq Me^{-\delta(i+j)},$$

then the first problem has a unique solution $\chi_\omega$, where $\omega$ is a measurable subset of $\Omega$ of Lebesgue measure $L|\Omega|$. Moreover,

- $\omega$ has a finite number of connected components,
- if $\Omega$ has a symmetry hyperplane, then $\omega$ enjoys the same symmetry property.

- For instance: ok if $y^0$ and $y^1$ are analytic.
- If $y^0$ and $y^1$ have $N$ nonzero coefficients, then the optimal set $\omega$ has at most $f(N)$ connected components (where the function $f$ can be characterized).
- The result can be generalized with quasi-analyticity:
  (see S. Mandelbrojt, Quasi-analycité des séries de Fourier)
- There exist $C^\infty$ data $(y^0, y^1)$ for which the optimal set $\omega$ has a fractal structure.
- Initial data $(y^0, y^1)$ for which $\omega$ is not unique can be characterized.
Solving of the second problem

\[ \sup_{\substack{\omega \subset \Omega \mid |\omega| = L|\Omega|}} C_T(\omega) = \sup_{\substack{\omega \subset \Omega \mid |\omega| = L|\Omega|}} \inf_{(a_j^2 + b_j^2) = 1} \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_{\omega} \phi_i(x) \phi_j(x) \, dx \]

We do not know how to handle this problem in general because of the crossed terms. If we remove the crossed terms then the second problem is

\[ \sup_{\substack{\omega \subset \Omega \mid |\omega| = L|\Omega|}} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx \]

There are two ways of getting rid of the crossed terms.
Solving of the second problem

\[
\sup_{\omega \subset \Omega} \frac{C_T(\omega)}{|\omega|=L|\Omega|} = \sup_{\omega \subset \Omega} \inf_{|\omega|=L|\Omega|} \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_\omega \phi_i(x) \phi_j(x) \, dx
\]

We do not know how to handle this problem in general because of the crossed terms. If we remove the crossed terms then the second problem is

\[
\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx
\]

There are two ways of getting rid of the crossed terms. **First way**: we rather consider the problem

\[
\sup_{\omega \subset \Omega} \lim_{T \to +\infty} \frac{C_T(\omega)}{T}
\]

**Lemma**

\[
\lim_{T \to +\infty} \sup_{\omega \subset \Omega} \frac{C_T(\omega)}{|\omega|=L|\Omega|} = \sup_{\omega \subset \Omega} \lim_{T \to +\infty} \frac{C_T(\omega)}{T} = \sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 \, dx
\]
Solving of the second problem

\[
\sup_{|\omega| = L|\Omega|} C_T(\omega) = \sup_{|\omega| = L|\Omega|} \inf_{\sum (a_j^2 + b_j^2) = 1} \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_{\omega} \phi_i(x) \phi_j(x) \, dx
\]

We do not know how to handle this problem in general because of the crossed terms. If we remove the crossed terms then the second problem is

\[
\sup_{|\omega| = L|\Omega|} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx
\]

There are two ways of getting rid of the crossed terms.

**Second way:** we consider the observability inequality

\[
C_{T, \text{rand}}(\omega) \| (y^0, y^1) \|_{L^2 \times H^{-1}}^2 \leq \mathbb{E} \left( \int_0^T \int_{\omega} y(t, x)^2 \, dx \, dt \right)
\]

in a probabilistic sense.

Then crossed terms disappear (see Burq-Tzvetkov, Invent. Math. 2008).
Solving of the second problem

\[ \sup_{\omega \subset \Omega} C_T(\omega) = \sup_{|\omega| = L|\Omega|} \inf_{\omega \subset \Omega} \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_{\omega} \phi_i(x) \phi_j(x) \, dx \]

We do not know how to handle this problem in general because of the crossed terms. If we remove the crossed terms then the second problem is

\[ \sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx \]

Remark 1:
This is an energy concentration criterion.

Remark 2:
The general problem with crossed terms is related with the (open) question of the existence of an optimal constant in Ingham’s inequality.
Solving of the second problem

Second problem

\[
\sup_{\omega \subset \Omega} \inf_{|\omega| = L|\Omega|} \int_{\Omega} \chi_\omega(x) \phi_j^2(x) \, dx
\]

1. Convexification procedure

\[
\overline{U}_L = \{ a \in L^\infty(\Omega, (0, 1)) \mid \int_{\Omega} a(x) \, dx = L|\Omega| \}.
\]

\[
\rightarrow \sup_{a \in \overline{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j^2(x) \, dx
\]

A priori:

\[
\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j^2(x) \, dx \leq \sup_{a \in \overline{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j^2(x) \, dx.
\]
Moreover, under the assumption

(weak Quantum Ergodicity) Assumption

There exists a subsequence such that $\phi_j^2 \rightharpoonup \frac{1}{|\Omega|}$ in weak star $L^\infty$ topology.

we have

$$\sup_{a \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j^2(x) \, dx = L$$

(reached with $a \equiv L$)

Remarks:

- It is true in 1D, since $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$ on $\Omega = [0, \pi]$.

Moreover, this relaxed problem has an infinite number of solutions, given by

$$a(x) = L + \sum_j (a_j \cos(2jx) + b_j \sin(2jx)) \text{ with } a_j \leq 0$$

(and with $|a_j|$ and $|b_j|$ small enough so that $0 \leq a(\cdot) \leq 1$).
Solving of the second problem

Moreover, under the assumption

(weak Quantum Ergodicity) Assumption

There exists a subsequence such that \( \phi_j^2 \rightarrow \frac{1}{|\Omega|} \) in weak star \( L^\infty \) topology.

we have

\[
\sup_{a \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j^2(x) \, dx = L
\]

(reached with \( a \equiv L \))

Remarks :

- In multi-D: it is true under ergodicity assumptions:

If \( \Omega \) is an ergodic billiard with \( W^{2,\infty} \) boundary then \( \phi_j^2 \rightarrow \frac{1}{|\Omega|} \) in weak star \( L^\infty \) for a subset of indices of density 1.


(see also Shnirelman, Burq-Zworski, Colin de Verdière, etc)
2. Gap or no-gap?

A priori, under the weak QE assumption:

\[
\sup_{\omega \subset \Omega} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j^2(x) \, dx \leq \sup_{a \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j^2(x) \, dx = L.
\]

Remarks in 1D:

- Note that, for every \( \omega \), \( \frac{2}{\pi} \int_{\omega} \sin^2(jx) \, dx \to L \) as \( j \to +\infty \).
- No lower semi-continuity property of the criterion.
- With \( \omega_N = \bigcup_{k=1}^{N} \left[ \frac{k\pi}{N+1} - \frac{L\pi}{2N}, \frac{k\pi}{N+1} + \frac{L\pi}{2N} \right] \), one has \( \chi_{\omega_N} \to L \) but

\[
\lim_{N \to +\infty} \inf_{j \in \mathbb{N}^*} \frac{2}{\pi} \int_{\omega_N} \sin^2(jx) \, dx < L.
\]
Solving of the second problem

(Quantum Unique Ergodicity) Assumption

We assume that \( \phi_j^2 \rightarrow \frac{1}{|\Omega|} \) in weak star \( L^\infty \) topology, as \( j \rightarrow +\infty \).

(i.e. the whole sequence converges to the Liouville measure)

Theorem

Under the QUE assumption, there is no gap, that is:

\[
\sup_{\chi \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega \chi \phi_j(x)^2 \, dx = \sup_{a \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega a(x) \phi_j(x)^2 \, dx = L.
\]

Remark: it holds also true e.g. in a square domain \( \Omega \), for which however QUE is not satisfied.
Solving of the second problem

(Quantum Unique Ergodicity) Assumption

We assume that $\phi_j^2 \rightharpoonup \frac{1}{|\Omega|}$ in weak star $L^\infty$ topology, as $j \to +\infty$. (i.e. the whole sequence converges to the Liouville measure)

Comments on this assumption:

- It is true in 1D, since $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$ on $\Omega = [0, \pi]$.
- Quantum Unique Ergodicity property (QUE) in multi-D:
    If $\Omega$ is an ergodic billiard with $W^{2,\infty}$ boundary then $\phi_j^2 \rightharpoonup \frac{1}{|\Omega|}$ in weak star $L^\infty$ for a subset of indices of density 1.
  - Strictly convex billiards sufficiently regular are not ergodic (Lazutkin, 1973). Rational polygonal billiards are not ergodic. Generic polygonal billiards are ergodic (Kerckhoff-Masur-Smillie, Ann. Math. '86).
  - There exist some convex sets $\Omega$ (stadium shaped) that satisfy QE but not QUE (Hassell, Ann. Math. 2010).
  - QUE conjecture (Rudnick-Sarnak 1994): every compact manifold having negative sectional curvature satisfies QUE.
Solving of the second problem

(Quantum Unique Ergodicity) Assumption

We assume that $\phi_j^2 \to \frac{1}{|\Omega|}$ in weak star $L^\infty$ topology, as $j \to +\infty$.

(i.e. the whole sequence converges to the Liouville measure)

Hence in general this assumption is related with ergodic / concentration / entropy properties of eigenfunctions.

See Shnirelman, Sarnak, Bourgain-Lindenstrauss, Colin de Verdière, Anantharaman, Nonenmacher, De Bièvre,...

If this assumption fails, we may have scars:
energy concentration phenomena
(there can be exceptional subsequences converging to other invariant measures, like, for instance, measures carried by closed geodesics: scars)
Solving of the second problem

(Quantum Unique Ergodicity) Assumption

We assume that \( \phi_j^2 \rightarrow \frac{1}{|\Omega|} \) in weak star \( L^\infty \) topology, as \( j \rightarrow +\infty \).

(i.e. the whole sequence converges to the Liouville measure)
Solving of the second problem

(Quantum Unique Ergodicity) Assumption

We assume that $\phi_j^2 \rightarrow \frac{1}{|\Omega|}$ in weak star $L^\infty$ topology, as $j \rightarrow +\infty$.

(i.e. the whole sequence converges to the Liouville measure)

Come back to the theorem:

Under QUE, there is no gap, that is :

$$\sup_{\chi \omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} \chi_\omega(x) \phi_j(x)^2 \, dx = \sup_{a \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j(x)^2 \, dx = L.$$ 

Moreover:

We are able to prove that, for certain sets $\Omega$, the second problem does not have any solution (i.e., the supremum is not reached). We conjecture that this property is generic.

Remark

QUE is not necessary. Example : 2D square.
Solving of the second problem

Last remark:

The proof of this no-gap result is based on a quite technical homogenization-like procedure. In dimension one, it happens that it is equivalent to the following harmonic analysis result:

Let $\mathcal{F}$ the set of functions

$$f(x) = L + \sum_{j=1}^{+\infty}(a_j \cos(2jx) + b_j \sin(2jx)), \quad \text{with } a_j \leq 0 \quad \forall j \in \mathbb{N}^*. $$

Then:

$$d(\mathcal{F}, \mathcal{U}_L) = 0$$

but there is no $\chi_\omega \in \mathcal{F}$.

(where $\mathcal{U}_L = \{\chi_\omega \mid \omega \subset [0, \pi], |\omega| = L\pi\}$)

E. Trélat

Optimal observation for wave equations
Since the second problem may have no solution, it makes sense to consider as in


a truncated version of the second problem:

\[
\sup_{\substack{\omega \subset \Omega \\
|\omega| = 2L|\Omega|}} \min_{1 \leq j \leq N} \int_{\omega} \phi_j^2(x) \, dx
\]
Truncated version of the second problem

\[
\sup_{\omega \subset \Omega \atop |\omega| = \frac{L}{2} |\Omega|} \min_{1 \leq j \leq N} \int_{\omega} \phi_j^2(x) \, dx
\]

**Theorem**

The problem has a unique solution \( \omega^N \).
Moreover, \( \omega^N \) has a finite number of connected components.
If \( \Omega \) has a symmetry hyperplane, then \( \omega^N \) enjoys the same symmetry property.
Theorem, specific to the 1D case

$\omega^N$ is symmetric with respect to $\pi/2$, is the union of at most $N$ intervals, and:

there exists $L_N \in (0, 1]$ such that, for every $L \in (0, L_N]$,

$$
\int_{\omega^N} \sin^2 x \, dx = \int_{\omega^N} \sin^2 (2x) \, dx = \cdots = \int_{\omega^N} \sin^2 (N x) \, dx.
$$

- Equality of the criteria $\Rightarrow$ the optimal domain $\omega^N$ concentrates around the points $\frac{k\pi}{N+1}$, $k = 1, \ldots, N$.

- Spillover phenomenon: the best domain $\omega^N$ for the $N$ first modes is the worst possible for the $N+1$ first modes.
Next issues (ongoing work with Y. Privat and E. Zuazua)

- Same results for Schrödinger equations.
- Same kind of analysis for the optimal design of the (HUM) control domain.
  
  In particular, for the first problem: complete characterization of all initial data for which
  - there exists an optimal set with a finite number of components
  - there exists an optimal set of Cantor type
  - there exists no optimal set (relaxation phenomenon)

- Relations between shape optimization and ergodicity properties.
- Consider other kinds of spectral criteria permitting to avoid the spillover phenomenon.
- Investigation of other equations such as the heat equation.
- Discretization issues: do the numerical optimal designs converge to the continuous optimal design as the mesh size tends to 0?
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