Journée MMSN - Ecole Polytechnique - CMAP

Large time-step and asymptotic-preserving numerical schemes for gaz dynamics with sources

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Introduction

Motivation : this work is motivated by the study of two-phase flows involved in nuclear reactors in nominal, incidental or accidental conditions

Several models and approaches :

- **"Micro"-scale models** : fine description of liquid/vapor interface topologies
- **"Macro"-scale models** : two-phase flow described as a mixture at thermodynamical equilibrium
- **"Middle"-scale models** : the so-called bi-fluid approach, takes into account desequilibrium between both phases

We are interested in the numerical approximation of one particular bi-fluid averaged model, the so-called 7-**equation or Baer-Nunziato like model**

The 7-equation model

In one space dimension, the model reads

$$\begin{cases} \dfrac{\partial \alpha_k}{\partial t} + u_I \dfrac{\partial \alpha_k}{\partial x} = \Theta(p_k - p_l), \\[2ex] \dfrac{\partial}{\partial t}(\alpha_k \varrho_k) + \dfrac{\partial}{\partial x}(\alpha_k \varrho_k u_k) = 0, \\[2ex] \dfrac{\partial}{\partial t}(\alpha_k \varrho_k u_k) + \dfrac{\partial}{\partial x}(\alpha_k(\varrho_k u_k^2 + p_k)) - p_I \dfrac{\partial \alpha_k}{\partial x} = \alpha_k \varrho_k g - \Lambda(u_k - u_l), \\[2ex] \dfrac{\partial}{\partial t}(\alpha_k \varrho_k e_k) + \dfrac{\partial}{\partial x}(\alpha_k(\varrho_k e_k + p_k) u_k) - p_I u_I \dfrac{\partial \alpha_k}{\partial x} = \alpha_k \varrho_k g u_k - p_I \Theta(p_k - p_l) - u_I \Lambda(u_k - u_l) \end{cases}$$

with $\alpha_1 + \alpha_2 = 1$

u_I, p_I : interfacial velocity and pressure (to be precised)

We note that the system is **nonconservative**, with short form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{U}) + \mathbf{B}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{S}(\mathbf{U})$$

THE GAS-DYNAMICS EQUATIONS WITH FRICTION AND GRAVITY TERMS

$\longrightarrow$ As a first-step, we consider the following model

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0, \\ \partial_t \varrho u + \partial_x (\varrho u^2 + p) = \varrho g - \varrho \alpha u, \\ \partial_t (\varrho E) + \partial_x (\varrho E u + p u) = \varrho u g - \varrho \alpha u^2 \end{cases}$$

where g is the gravity constant and α is the friction coefficient.

$\longrightarrow$ In many codes, the very complex geometry of the core of a nuclear code is modelled by means of a subgrid model

$\longrightarrow$ Here we consider that the core is modelled as a porous medium, and that α is related to the friction between the fluid and the fuel rods zirconium-clads.

Then, α models the wall-friction influence of the channels upon the fluid

Remark on α and the source term stiffness.
In practice, the magnitude of the friction coefficient α lies between 0.5 and 1.0, *which is by no mean a large value, but...*

The gas-dynamics equations with friction and gravity terms

$\longrightarrow$ One may be interested in long-time stationnary or nearly-stationnary flow profiles. In dimensionless form, α is then multiplied by a large characteristic time $\bar{t}$:

$$\alpha \approx \frac{\alpha}{\epsilon}, \quad \epsilon \ll 1,$$

$\longrightarrow$ The spatial discretization Δx may be very coarse. Since the product $\alpha \Delta x$ will play an important role in the consistency errors, it amounts to consider here again

$$\alpha \approx \frac{\alpha}{\epsilon}, \quad \epsilon \ll 1,$$

$\longrightarrow$ Other applications more naturally lead to large friction coefficients...

The gas-dynamics equations with friction and gravity terms

We are thus interested in

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0, \\ \partial_t \varrho u + \partial_x(\varrho u^2 + p) = \varrho g - \varrho \dfrac{\alpha}{\epsilon} u, \\ \partial_t(\varrho E) + \partial_x(\varrho E u + pu) = \varrho u g - \varrho \dfrac{\alpha}{\epsilon} u^2 \end{cases}$$

for a small parameter $\epsilon \ll 1$. Then we have $u = \mathcal{O}(\epsilon)$

$\longrightarrow$ The limit $\epsilon \to 0$ can be considered as a model worst-case scenario for testing the accuracy of the method in the presence of friction source term

Asymptotic analysis. Set $u = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2)$, and $e = E - u^2/2$.
The model reads

$$\begin{cases} u_0 = 0, \\ \partial_t \varrho + \epsilon \partial_x \varrho u_1 = \mathcal{O}(\epsilon^2) \\ \varrho u_1 = \dfrac{1}{\alpha}(\varrho g - \partial_x p) + \mathcal{O}(\epsilon) \\ \partial_t(\varrho e) + \epsilon \partial_x(\varrho e u_1 + p u_1) = \epsilon \varrho u_1 (g - \alpha u_1) + \mathcal{O}(\epsilon^2) \end{cases}$$

The gas-dynamics equations with friction and gravity terms

We are thus interested in

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0, \\ \partial_t \varrho u + \partial_x (\varrho u^2 + p) = \varrho g - \varrho \dfrac{\alpha}{\epsilon} u, \\ \partial_t (\varrho E) + \partial_x (\varrho E u + p u) = \varrho u g - \varrho \dfrac{\alpha}{\epsilon} u^2 \end{cases}$$

for a small parameter $\epsilon \ll 1$. Then we have $u = \mathcal{O}(\epsilon)$

$\longrightarrow$ The limit $\epsilon \to 0$ can be considered as a model worst-case scenario for testing the accuracy of the method in the presence of friction source term

Asymptotic analysis. Set $u = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2)$, $t = s/\epsilon$ and $e = E - u^2/2$. The long-time behaviour of the solutions is given by

$$\begin{cases} u_0 = 0, \\ \partial_s \varrho + \partial_x \varrho u_1 = \mathcal{O}(\epsilon), \\ \varrho u_1 = \dfrac{1}{\alpha}(\varrho g - \partial_x p) + \mathcal{O}(\epsilon), \\ \partial_s (\varrho e) + \partial_x (\varrho e u_1 + p u_1) = \varrho u_1 (g - \alpha u_1) + \mathcal{O}(\epsilon) \end{cases}$$

See Hsiao-Liu, Nishihara, Junca-Rascle, Lin-Coulombel, Coulombel-Goudon, Marcati-Milani... for rigorous proofs. See Formal derivation for formal derivation

The gas-dynamics equations with friction and gravity terms

Let us briefly recall that :

- Eigenvalues are given by

$$u - c \quad u \quad u + c$$

  where c is the sound speed
- Pressure laws may be strongly non linear, even tabulated, which makes difficult the resolution

- Time-step CFL restrictions are naturally based on acoustic waves in Godunov-type schemes

$$\max_{\mathbf{u}}(|u \pm c|, |u|) \frac{\Delta t}{\Delta x} \leq \frac{1}{2}$$

- Acoustic waves are not expected to be predominant here (too bad !), since flows are *subsonic* and/or with *low Mach number* in nuclear reactors

Objectives

Our objective is to propose a **numerical scheme** :

- able to deal with any pressure law p
- stable under a more adapted time-step CFL restriction based on u and that does not depend on ϵ

$$\max_{\mathbf{u}}(|u|)\,\frac{\Delta t}{\Delta x} \leq \frac{1}{2}$$

- and asymptotic-preserving

Definition of asymptotic-preserving

Definition of asymptotic-preserving scheme. Let us denote

- M^ϵ the initial model
- M^0 the limit model
- $S^\epsilon_{\Delta t,\Delta x}$ the proposed numerical scheme
- $S^0_{\Delta t,\Delta x}$ the limit numerical scheme

With little abuse in the notations, S^ϵ is said to be asymptotic-preserving if

- for all $\epsilon > 0$, $S^\epsilon_{\Delta t,\Delta x}$ is stable[1] and consistent with M^ϵ : $\lim_{\Delta t,\Delta x\to 0} S^\epsilon_{\Delta t,\Delta x} = M^\epsilon$
- $S^0_{\Delta t,\Delta x}$ is stable and consistent with M^0 : $\lim_{\Delta t,\Delta x\to 0} S^0_{\Delta t,\Delta x} = M^0$

In other words, asymptotic-preserving property means **order of limits interchange** property

$$\lim_{\epsilon\to 0}\lim_{\Delta t,\Delta x\to 0} S^\epsilon_{\Delta t,\Delta x} = \lim_{\Delta t,\Delta x\to 0}\lim_{\epsilon\to 0} S^\epsilon_{\Delta t,\Delta x}$$

[1] independently of $\epsilon > 0$, in some sense to be precised...

How to reach these objectives ?

How to get the expected CFL restriction ?
- implicit treatment of the acoustic waves $u \pm c$
- explicit treatment of the transport waves u (predominant, we want to keep accuracy)
$\rightarrow$ Lagrange-Projection strategy
(**see** *Coquel, Nguyen, Postel, Tran*, Math. Comp 2010)

How to deal with any (possibly strongly nonlinear) pressure law p ?
- overcome the non linearities, "linearization"
$\rightarrow$ Relaxation strategy
(**see** *Chalons, Coquel*, Numer. Math. 2005)

How to get the asymptotic-preserving (AP) property ?
- upwind and implicit treatment of the source
$\rightarrow$ Notion of consistency with the integral form of the full model
(**see** *Gallice*, Numer. Math. 2003 **and**
Chalons, Coquel, Godlewski, Raviart, Seguin, M3AS 2010)

Lagrange-Projection strategy for the gas-dynamics equations

Let us first focus on

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0 \\ \partial_t \varrho u + \partial_x(\varrho u^2 + p) = 0 \\ \partial_t(\varrho E) + \partial_x(\varrho E u + pu) = 0 \end{cases}$$

Using chain rule arguments, we also have

$$\begin{cases} \partial_t \varrho + u\partial_x \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + u\partial_x \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + u\partial_x \varrho E + \varrho E \partial_x u + \partial_x pu = 0 \end{cases}$$

so that splitting the transport part leads to

$$\underbrace{\begin{cases} \partial_t \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + \varrho E \partial_x u + \partial_x pu = 0 \end{cases}}_{\text{Lagrangian-step}} \qquad \underbrace{\begin{cases} \partial_t \varrho + u\partial_x \varrho = 0 \\ \partial_t \varrho u + u\partial_x \varrho u = 0 \\ \partial_t \varrho E + u\partial_x \varrho E = 0 \end{cases}}_{\text{Transport-step}}$$

Lagrange-Projection approach

The Lagrangian-step

$$\begin{cases} \partial_t \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + \varrho E \partial_x u + \partial_x p u = 0 \end{cases}$$

also writes

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m p u = 0 \end{cases}$$

with $\tau = 1/\varrho$ and $\tau \partial_x = \partial_m$.

- Eigenvalues are given by $-\rho c,\ 0,\ \rho c$
- Usual CFL conditions for time-explicit schemes write

$$\frac{\Delta t}{\Delta x} \max(\rho c) \leq \frac{1}{2}$$

The idea is to propose a time-implicit scheme to avoid this time-step restriction
Question : How to do that in a very cheap way and for any pressure law ?

Lagrange-Projection approach

The Transport-step

$$\left\{\begin{array}{l}\partial_t\varrho + u\partial_x\varrho = 0\\ \partial_t\varrho u + u\partial_x\varrho u = 0\\ \partial_t\varrho E + u\partial_x\varrho E = 0\end{array}\right.$$

- Eigenvalues are given by u
- Usual CFL conditions for time-explicit schemes write

$$\frac{\Delta t}{\Delta x}\max(|u|) \leq \frac{1}{2}$$

The idea is then to propose a standard time-explicit scheme to keep accuracy on the (slow) contact waves

Generalities

The gas dynamics in Lagrangian coordinates :

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m pu = 0 \end{cases}$$

with $p = p(\tau, e)$ and

$$e = E - \frac{1}{2}u^2$$

Due to the nonlinearities of p, the Riemann problem is difficult to solve.

The relaxation strategy :

- **Idea :** to deal with a larger but simpler system
- **Design principle :** to understand $p(\tau, e)$ as a new unknown that we denote Π

GENERALITIES

The gas dynamics in Lagrangian coordinates :

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m pu = 0 \end{cases}$$

The relaxation system :

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m \Pi = 0 \\ \partial_t E + \partial_m \Pi u = 0 \\ \partial_t \Pi + a^2 \partial_m u = \lambda(p - \Pi) \end{cases}$$

Recall that

$$\partial_t p + \rho^2 c^2 \partial_m u = 0$$

At least formally, observe that

$$\lim_{\lambda \to +\infty} \Pi = p \quad (\text{if} \quad a > \rho c(\tau, e))$$

(**see e.g.** *Chalons*, *Coulombel*, Analysis and Applications 2008 **for a rigorous proof**)

Properties

The relaxation system :

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m \Pi = 0 \\ \partial_t E + \partial_m \Pi u = 0 \\ \partial_t \Pi + a^2 \partial_m u = \lambda (p - \Pi) \end{cases}$$

This system is **strictly hyperbolic** with the following eigenvalues

$$-a < 0 < a$$

The characteristic fields are all **linearly degenerate** (the waves behave more or less as linear waves)

The Riemann problem is explicitly solved

RIEMANN SOLUTION FOR THE RELAXATION SYSTEM

The intermediate states are simply found thanks to the Rankine-Hugoniot conditions across each waves

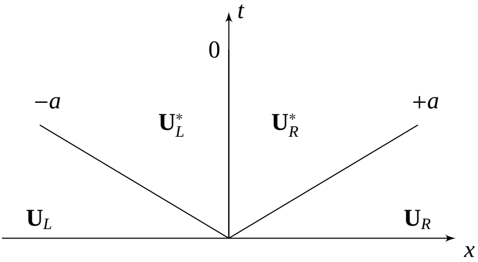


FIG.: Approximate Riemann solver - general structure

The numerical scheme

As a consequence, the numerical strategy for solving the equilibrium system

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m pu = 0 \end{cases}$$

consists **at each time step** in

- the classical Godunov scheme for the relaxation system

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m \Pi = 0 \\ \partial_t E + \partial_m \Pi u = 0 \\ \partial_t \Pi + a^2 \partial_m u = 0 \end{cases}$$

- with initial data at equilibrium that is such that $\Pi = p$

The time-explicit Godunov scheme for the relaxation system

The relaxation system writes

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m \Pi = 0 \\ \partial_t \Pi + a^2 \partial_m u = 0 \\ \partial_t E + \partial_m \Pi u = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t (\Pi + au) + a\partial_m (\Pi + au) = 0 \\ \partial_t (\Pi - au) - a\partial_m (\Pi + au) = 0 \\ \partial_t E + \partial_m \Pi u = 0 \end{cases}$$

or equivalently

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t w^+ + a\partial_m w^+ = 0 \\ \partial_t w^- - a\partial_m w^- = 0 \\ \partial_t E + \partial_m \Pi u = 0 \end{cases}$$

with

$$u = \frac{w^+ - w^-}{2a}, \quad \Pi = \frac{w^+ + w^-}{2}, \quad w^\pm = \Pi \pm au$$

The time-explicit Godunov scheme for the relaxation system

The relaxation system writes

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t w^+ + a\partial_m w^+ = 0 \\ \partial_t w^- - a\partial_m w^- = 0 \\ \partial_t E + \partial_m \Pi u = 0 \end{cases}$$

with

$$u = \frac{w^+ - w^-}{2a}, \quad \Pi = \frac{w^+ + w^-}{2}, \quad w^\pm = \Pi \pm au$$

The time-explicit Godunov scheme for the relaxation system writes

$$\begin{cases} \overline{w}_j^+ = w_j^+ - a\frac{\Delta t}{\Delta m}(w_j^+ - w_{j-1}^+) \\ \overline{w}_j^- = w_j^- + a\frac{\Delta t}{\Delta m}(w_{j+1}^- - w_j^-) \\ \overline{\tau}_j = \tau_j + \frac{\Delta t}{\Delta m}(u_{j+1/2} - u_{j-1/2}) \\ \overline{E}_j = E_j - \frac{\Delta t}{\Delta m}(\Pi_{j+1/2}u_{j+1/2} - \Pi_{j+1/2}u_{j-1/2}) \end{cases}$$

with

$$u_{j+1/2} = \frac{w_j^+ - w_{j+1}^-}{2a}, \quad \Pi_{j+1/2} = \frac{w_j^+ + w_{j+1}^-}{2}, \quad w_j^\pm = \Pi_j \pm au_j$$

The time-explicit Godunov scheme for the relaxation system

The time-explicit Godunov scheme for the relaxation system writes

$$\begin{cases} \overline{w}_j^+ = w_j^+ - a\frac{\Delta t}{\Delta m}(w_j^+ - w_{j-1}^+) \\ \overline{w}_j^- = w_j^- - a\frac{\Delta t}{\Delta m}(w_{j+1}^- - w_j^-) \\ \overline{\tau}_j = \tau_j + \frac{\Delta t}{\Delta m}(u_{j+1/2} - u_{j-1/2}) \\ \overline{E}_j = E_j - \frac{\Delta t}{\Delta m}(\Pi_{j+1/2}u_{j+1/2} - \Pi_{j+1/2}u_{j-1/2}) \end{cases}$$

with

$$u_{j+1/2} = \frac{w_j^+ - w_{j+1}^-}{2a}, \quad \Pi_{j+1/2} = \frac{w_j^+ + w_{j+1}^-}{2}, \quad w_j^\pm = \Pi_j \pm a u_j$$

Remarks.

- This scheme applies for any pressure law !
- The CFL condition for this time-explicit schemes write

$$\frac{\Delta t}{\Delta x} a \leq \frac{1}{2}$$

  that is, since $a \equiv \max(\rho c)$,

$$\frac{\Delta t}{\Delta x}\max(\rho c) \leq \frac{1}{2}.$$

- This scheme is stable and satisfies an entropy inequality provided that $a > \rho c$

(**see** *Chalons, Coquel*, Numer. Math. 2005)

THE GAS DYNAMICS IN LAGRANGIAN COORDINATES WITH FRICTION AND GRAVITY TERMS

$\longrightarrow$ Just for simplicity, we now focus on the barotropic case and drop the gravity
$\longrightarrow$ We propose to take into the source terms in the Lagrangian step

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = -\dfrac{\alpha}{\epsilon} u \end{cases}$$

Continuous asymptotic analysis. $u = 0 + \epsilon u_1 + \mathcal{O}(\epsilon^2), \quad t = s/\epsilon$

$$\begin{cases} \partial_s \tau - \partial_m u_1 = 0, \\ u_1 = -\dfrac{1}{\alpha} \partial_m p \end{cases}$$

Numerical asymptotic analysis. $u = 0 + \epsilon u_1 + \mathcal{O}(\epsilon^2), \quad t = s/\epsilon$

- usual splitting techniques do not work !
- the numerical flux $u_{j+1/2}$ should see α

Idea : include the source term in the approximate Riemann solver based on the previous relaxation system

Relaxation system with source terms

We consider the relaxation system

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m \Pi = -\dfrac{\alpha}{\epsilon} u \\ \partial_t \Pi + a^2 \partial_m u = 0 \end{cases}$$

and apply generalized (resp. classical) Rankine-Hugoniot relations across the stationary (resp. non stationary) waves

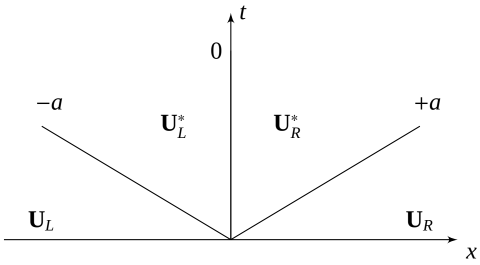


Fig.: Approximate Riemann solver - general structure

Consistency of Simple Approximate Riemann Solvers

Conservative system

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{U}) = 0$$

Consistency property with the integral form (Harten-Lax-Van Leer formalism)

$$\int_{-\Delta x/2}^{\Delta x/2} \mathbf{W}(x/\Delta t; \mathbf{U}_L, \mathbf{U}_R)dx = \frac{\Delta x}{2}(\mathbf{U}_L + \mathbf{U}_R) - \Delta t(\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L))$$

Conservative system with sources

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U})$$

Consistency property with the integral form (see also Gallice's formalism)

$$\int_{-\Delta x/2}^{\Delta x/2} \mathbf{W}(x/\Delta t; \mathbf{U}_L, \mathbf{U}_R)dx = \frac{\Delta x}{2}(\mathbf{U}_L + \mathbf{U}_R) - \Delta t\Big(\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L)\Big) + \Delta t \Delta x\, \mathbf{S}(\mathbf{U}_L, \mathbf{U}_R)$$

with $\mathbf{S}(\mathbf{U},\mathbf{U}) = \mathbf{S}(\mathbf{U})$

Evaluating the six unknowns

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m \Pi = -\dfrac{\alpha}{\epsilon} u \\ \partial_t \Pi + a^2 \partial_m u = 0 \end{cases}$$

Consistency relations : 3 equations

$$\begin{cases} -a(\tau_L^* - \tau_L) + a(\tau_R - \tau_R^*) = u_L - u_R \\ -a(u_L^* - u_L) + a(u_R - u_R^*) = \Pi_R - \Pi_L + \dfrac{\alpha}{\epsilon} \Delta m \, \tilde{u} \\ -a(\Pi_L^* - \Pi_L) + a(\Pi_R - \Pi_R^*) = a^2 (u_R - u_L) \end{cases}$$

Mass conservation across each wave : 2 equations

$$\begin{cases} u_L - a\tau_L = u_L^* - a\tau_L^* \\ u_R + a\tau_R = u_R^* + a\tau_R^* \\ u_L^* = u_R^* =: u^* \end{cases}$$

Generalized Rankine-Hugoniot relation at the interface : 1 equation

$$\Pi_R^* - \Pi_L^* = -\frac{\alpha}{\epsilon} \Delta m \, u^*$$

The time-explicit Godunov scheme with sources

The time-explicit Godunov scheme writes

$$\begin{cases} \overline{w}_j^+ = w_j^+ - a\frac{\Delta t}{\Delta m}(w_j^+ - w_{j-1}^+) - \frac{\alpha}{\epsilon}a\Delta t\, u_{j-1/2}^* \\ \overline{w}_j^- = w_j^- - a\frac{\Delta t}{\Delta m}(w_{j+1}^- - w_j^-) + \frac{\alpha}{\epsilon}a\Delta t\, u_{j+1/2}^* \\ \overline{\tau}_j = \tau_j + \frac{\Delta t}{\Delta m}(u_{j+1/2}^* - u_{j-1/2}^*) \end{cases}$$

with $$\frac{u_{j+1/2}^*}{\epsilon} = \frac{(w_j^+ - w_{j+1}^-)}{2a\epsilon + \alpha\Delta m} = \frac{2a}{2a\epsilon + \alpha\Delta m}\Big(\frac{u_j + u_{j+1}}{2} - \frac{\Pi_{j+1} - \Pi_j}{2a}\Big)$$

Numerical asymptotic analysis. $t = s/\epsilon$

- Multiply the first two equations by ϵ and let $\epsilon \to 0$: $u_j^n = 0$
- $\epsilon \to 0$ in the last equation :

$$\begin{cases} \overline{\tau}_j = \tau_j + \dfrac{\Delta s}{\Delta m}(u_{1,j+1/2} - u_{1,j-1/2}), \\ u_{1,j+1/2} = -\dfrac{1}{\alpha}\dfrac{\Pi_{j+1}^n - \Pi_j^n}{\Delta m} \end{cases}$$

which is consistent with

$$\begin{cases} \partial_s \tau - \partial_m u_1 = 0, \\ u_1 = -\dfrac{1}{\alpha}\partial_m p \end{cases}$$

The classical explicit-implicit splitting operator scheme

The classical explicit-implicit splitting operator scheme

$$\begin{cases} \overline{w}_j^+ = w_j^+ - a\frac{\Delta t}{\Delta m}(w_j^+ - w_{j-1}^+) - \frac{\alpha}{\epsilon}a\Delta t\,\overline{u}_j \\ \overline{w}_j^- = w_j^- - a\frac{\Delta t}{\Delta m}(w_{j+1}^- - w_j^-) + \frac{\alpha}{\epsilon}a\Delta t\,\overline{u}_j \\ \overline{\tau}_j = \tau_j + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \end{cases}$$

with $$\frac{u^*_{j+1/2}}{\epsilon} = \frac{(w_j^+ - w_{j+1}^-)}{2a\epsilon} = \frac{1}{\epsilon}\Big(\frac{u_j + u_{j+1}}{2} - \Delta m\frac{\Pi_{j+1} - \Pi_j}{\Delta m 2a}\Big)$$

Numerical asymptotic analysis. $u_j = u_j^{(0)} + \epsilon u_j^{(1)} + \mathcal{O}(\epsilon^2)$, $t = s/\epsilon$

- Multiply the first two equations by ϵ and let $\epsilon \to 0$: $u_j^{(0)} = 0$
- Make the difference of the first two equations and let $\epsilon \to 0$:

$$\frac{u_j^{(1)} + u_{j+1}^{(1)}}{2} \to v_{j+1/2} \approx -\frac{1}{\alpha}\frac{\Pi^n_{j+1} - \Pi^n_j}{\Delta m}$$

- Let then $\epsilon \to 0$ in the last equation :

$$\begin{cases} \overline{\tau}_j = \tau_j + \dfrac{\Delta s}{\Delta m}(u_{1,j+1/2} - u_{1,j-1/2}), \\ u_{1,j+1/2} = v_{j+1/2} - \dfrac{\Pi_{j+1} - \Pi_j}{2a\epsilon} = v_{j+1/2} + \mathcal{O}(\dfrac{\Delta m}{\epsilon}) \end{cases}$$

which is clearly not consistent with $\begin{cases} \partial_s \tau - \partial_m u_1 = 0, \\ u_1 = -\dfrac{1}{\alpha}\partial_m p \end{cases}$

The time-explicit Godunov scheme with sources

The time-explicit Godunov scheme writes

$$\begin{cases} \overline{w}_j^+ = w_j^+ - a\frac{\Delta t}{\Delta m}(w_j^+ - w_{j-1}^+) - \frac{\alpha}{\epsilon} a\Delta t\, u^*_{j-1/2} \\ \overline{w}_j^- = w_j^- - a\frac{\Delta t}{\Delta m}(w_{j+1}^- - w_j^-) + \frac{\alpha}{\epsilon} a\Delta t\, u^*_{j+1/2} \\ \overline{\tau}_j = \tau_j + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \end{cases}$$

$$\text{with} \qquad \frac{u^*_{j+1/2}}{\epsilon} = \frac{(w_j^+ - w_{j+1}^-)}{2a\epsilon + \alpha\Delta m}$$

About the CFL condition of such an explicit scheme.

- The scheme is still based on the relaxation approximate Riemann solver, then

$$\frac{\Delta t}{\Delta x} a \le \frac{1}{2}$$

- The change of variable $t = s/\epsilon$ gives

$$\frac{\Delta s}{\Delta x} a \le \frac{\epsilon}{2}$$

In the limit $\epsilon \to 0$, we get $\Delta s = 0$! ! ! **Not satisfying**

The time-explicit Godunov scheme with sources

The time-explicit Godunov scheme writes

$$\begin{cases} \overline{w}_j^+ = w_j^+ - a\frac{\Delta t}{\Delta m}(w_j^+ - w_{j-1}^+) - \frac{\alpha}{\epsilon} a\Delta t\, u_{j-1/2}^* \\ \overline{w}_j^- = w_j^- - a\frac{\Delta t}{\Delta m}(w_{j+1}^- - w_j^-) + \frac{\alpha}{\epsilon} a\Delta t\, u_{j+1/2}^* \\ \overline{\tau}_j = \tau_j + \frac{\Delta t}{\Delta m}(u_{j+1/2}^* - u_{j-1/2}^*) \end{cases}$$

$$\text{with} \qquad \frac{u_{j+1/2}^*}{\epsilon} = \frac{(w_j^+ - w_{j+1}^-)}{2a\epsilon + \alpha\Delta m}$$

About the CFL condition of such an explicit scheme.

- A possible cure is to implicit the centered part of the source term (see Gosse-Toscani) in order to get a CFL condition of the following form

$$\frac{a}{2a\epsilon + \alpha\Delta x}\frac{\Delta s}{\Delta x}a \leq \frac{\epsilon}{2},$$

  which gives in the limit $\epsilon \to 0$

$$\frac{a^2}{\alpha}\frac{\Delta s}{\Delta x^2} \leq \frac{\epsilon}{2}.$$

  Which is nothing but the classical parabolic time step restriction !
- Here, recall however that our objective is to get rid of any CFL restriction involving c. We then propose to implicit both the convective part and the source term as before (the rigorous proof of non linear stability is still open at this stage)

The time-implicit Godunov scheme for the relaxation system

The time-implicit Godunov scheme for the relaxation system writes

$$\left\{\begin{array}{l} \overline{w}_j^+ = w_j^+ - a\frac{\Delta t}{\Delta m}(\overline{w}_j^+ - \overline{w}_{j-1}^+) - \frac{\alpha}{\epsilon} a\Delta t\, \overline{u}_{j-1/2}^* \\ \overline{w}_j^- = w_j^- - a\frac{\Delta t}{\Delta m}(\overline{w}_{j+1}^- - \overline{w}_j^-) + \frac{\alpha}{\epsilon} a\Delta t\, \overline{u}_{j+1/2}^* \\ \overline{\tau}_j = \tau_j + \frac{\Delta t}{\Delta m}(\overline{u}_{j+1/2}^* - \overline{u}_{j-1/2}^*) \end{array}\right.$$

$$\text{with} \qquad \frac{\overline{u}_{j+1/2}^*}{\epsilon} = \frac{(\overline{w}_j^+ - \overline{w}_{j+1}^-)}{2a\epsilon + \alpha\Delta m}$$

Remarks.

- This scheme still applies for any pressure law !
- Updating w^+ and w^- (now coupled) amounts to solve a **pentadiagonal and diagonally dominant** system, and updating τ follows explicitly

THE TIME-IMPLICIT GODUNOV SCHEME FOR THE RELAXATION SYSTEM

The time-implicit Godunov scheme for the relaxation system writes

$$\begin{cases} \overline{w}_j^+ = w_j^+ - a\frac{\Delta t}{\Delta m}(\overline{w}_j^+ - \overline{w}_{j-1}^+) - \frac{\alpha}{\epsilon} a\Delta t\, \overline{u}_{j-1/2}^* \\ \overline{w}_j^- = w_j^- - a\frac{\Delta t}{\Delta m}(\overline{w}_{j+1}^- - \overline{w}_j^-) + \frac{\alpha}{\epsilon} a\Delta t\, \overline{u}_{j+1/2}^* \\ \overline{\tau}_j = \tau_j + \frac{\Delta t}{\Delta m}(\overline{u}_{j+1/2}^* - \overline{u}_{j-1/2}^*) \end{cases}$$

$$\text{with} \qquad \frac{\overline{u}_{j+1/2}^*}{\epsilon} = \frac{(\overline{w}_j^+ - \overline{w}_{j+1}^-)}{2a\epsilon + \alpha\Delta m}$$

Numerical asymptotic analysis. $t = s/\epsilon$

- Multiply the first two equations by ϵ and let $\epsilon \to 0$: $\overline{u}_j = 0$
- $\epsilon \to 0$ in the last equation :

$$\begin{cases} \overline{\tau}_j = \tau_j + \dfrac{\Delta s}{\Delta m}(\overline{u}_{1,j+1/2} - \overline{u}_{1,j-1/2}), \\ \overline{u}_{1,j+1/2} = -\dfrac{1}{\alpha}\dfrac{\overline{\Pi}_{j+1}^n - \overline{\Pi}_j^n}{\Delta m} \end{cases}$$

which is consistent with

$$\begin{cases} \partial_s \tau - \partial_m u_1 = 0, \\ u_1 = -\dfrac{1}{\alpha}\partial_m p \end{cases}$$

Properties and open questions

Properties

The **explicit or semi-implicit** Lagrange-Projection scheme with sources

- can be re-written as a conservative finite volume scheme
- is Asymptotic-Preserving
- can be extended to non linear friction terms with gravity, to the non barotropic case, and to the multi-fluid case

The **explicit** Lagrange-Projection scheme with sources

- is entropy satisfying

Open question

- Is the **semi-implicit** Lagrange-Projection scheme with sources entropy satisfying ?

Remark

- The Lagrangian part of the **explicit** scheme coincides with the one proposed in *Chalons C., Coquel F., Godlewski E., Raviart P.-A., Seguin N. Godunov-type schemes for hyperbolic systems with parameter dependent source. The case of Euler system with friction, M3AS (2010)*

Description

$\longrightarrow$ Perfect gas equation of state $p = (\gamma - 1)\rho e$ and

$$g = 9.81\,\mathrm{m}\cdot\mathrm{s}^{-2}, \quad \alpha = 10^6\,\mathrm{s}^{-1}, \quad \gamma = 1.4.$$

$\longrightarrow$ Initial condition

$$\begin{cases} (\rho, u, p) = (1.0, 0, 10000.0), & \text{if } x \in [0, 0.35] \cap [0.65, 1], \\ (\rho, u, p) = (2.0, 0, 26390.2), & \text{if } x \in [0.35, 0.65]. \end{cases}$$

$\longrightarrow$ Periodic boundary conditions

Sensitivity with respect to the space step for large friction

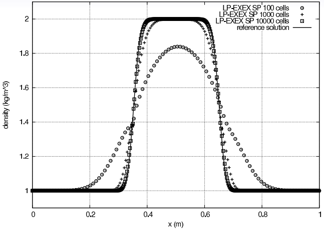


Fig.: Profile at time $t = 0.01$ s of the density obtained for a 100-cell, 1000-cell and 10 000-cell grid with the LP-EXEX SP scheme and the reference solution.

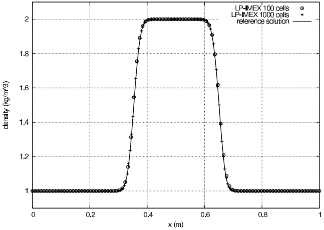


Fig.: Profile at time $t = 0.01$ s of the density obtained for a 100-cell and 1000-cell grid with the LP-IMEX scheme and the reference solution.

Sensitivity with respect to the space step for large friction

Tab.: Comparison of the relative errors between the approximated solutions obtained with both LP-EXEX SP and LP-IMEX schemes. The space domain is discretized with a 1 000-cell space discretization and $\Delta t = \frac{1}{\alpha}$ for both schemes.

numerical scheme	$\mathrm{err}(\rho, t = 0.01)$	$\mathrm{err}(u, t = 0.01)$	$\mathrm{err}(P, t = 0.01)$
LP-EXEX SP	1.686931×10^{-2}	6.858335×10^{-1}	2.539820×10^{-2}
LP-IMEX	3.959560×10^{-4}	1.195630×10^{-2}	5.635518×10^{-4}

Sensitivity with respect to the time step

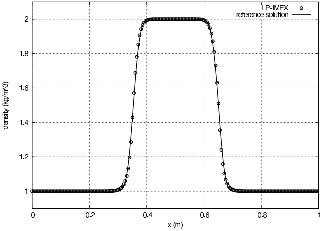


Fig.: Profile at time $t = 0.01\,$s of the density obtained for a 1000-cell grid with the LP-IMEX scheme and the reference solution.

Sensitivity with respect to the space step for large friction

Tab.: Comparison of the relative L^1-errors obtained with the LP-IMEX scheme for a 1 000-cell space discretization and two different Δt values.

numerical scheme	Δt	$\mathrm{err}(\rho, t = 0.01)$	$\mathrm{err}(u, t = 0.01)$	$\mathrm{err}(P, t = 0.01)$
LP-IMEX	$\frac{1}{\alpha}$	3.959560×10^{-4}	1.195630×10^{-2}	5.635518×10^{-4}
LP-IMEX	$\frac{1000}{\alpha}$	2.607495×10^{-3}	1.099137×10^{-1}	3.288768×10^{-3}

Conclusions and future works

Conclusions

- We have been interested in intermediate regimes (subsonic flows and coarse meshes) and possible stationary or nearly stationary flows

- The limit $\epsilon \to 0$ is seen as a worst-case scenario "only"

- AP strategies turn out to be sufficient strategies in order to lessen the numerical diffusion and get good numerical results even for coarse meshes (note that the proposed numerical results show the benefit even if u is not of order ϵ or the solution not stationary)

Future works

- 2D

- Low-Mach regimes

- Two-phase flow models

The time-explicit Godunov scheme for the transport step

The transport step of the Lagrange-Projection strategy writes

$$\begin{cases} \partial_t \rho + u\partial_x \rho = 0 \\ \partial_t(\rho u) + u\partial_x(\rho u) = 0 \\ \partial_t(\rho E) + u\partial_x(\rho E) = 0 \end{cases}$$

that is $\partial_t X + u\partial_x X = 0$, with $X = \rho, \rho u, \rho E$

The time-explicit Godunov scheme for this equation writes

- either (fully explicit approach)

$$X_j^{n+1} = \frac{\Delta t}{\Delta x} u^+_{j-1/2} \overline{X}_{j-1} + \frac{\Delta t}{\Delta x}\Big(1 + (u^-_{j+1/2} - u^+_{j-1/2})\Big)\overline{X}_j - \frac{\Delta t}{\Delta x} u^-_{j+1/2}\overline{X}_{j+1}$$

  with $u^+ = \max(u,0)$, $u^- = \min(u,0)$ and $u_{j+1/2} = \dfrac{w_j^+ - w_{j+1}^-}{2a}$

- or (semi-implicit approach)

$$X_j^{n+1} = \frac{\Delta t}{\Delta x} \overline{u}^+_{j-1/2} \overline{X}_{j-1} + \frac{\Delta t}{\Delta x}\Big(1 + (\overline{u}^-_{j+1/2} - \overline{u}^+_{j-1/2})\Big)\overline{X}_j - \frac{\Delta t}{\Delta x} \overline{u}^-_{j+1/2}\overline{X}_{j+1}$$

  with $u^+ = \max(u,0)$, $u^- = \min(u,0)$ and $\overline{u}_{j+1/2} = \dfrac{\overline{w}_j^+ - \overline{w}_{j+1}^-}{2a}$

The 7-equation model

In $1D$ and dimensionless form, the model reads

$$\begin{cases} \dfrac{\partial \alpha_k}{\partial t} + u_I \dfrac{\partial \alpha_k}{\partial x} = \Theta(p_k - p_l), \\[2ex] \dfrac{\partial}{\partial t}(\alpha_k \varrho_k) + \dfrac{\partial}{\partial x}(\alpha_k \varrho_k u_k) = 0, \\[2ex] \dfrac{\partial}{\partial t}(\alpha_k \varrho_k u_k) + \dfrac{\partial}{\partial x}(\alpha_k(\varrho_k u_k^2 + p_k)) - p_I \dfrac{\partial \alpha_k}{\partial x} = \alpha_k \varrho_k g - \Lambda(u_k - u_l), \\[2ex] \dfrac{\partial}{\partial t}(\alpha_k \varrho_k e_k) + \dfrac{\partial}{\partial x}(\alpha_k(\varrho_k e_k + p_k) u_k) - p_I u_I \dfrac{\partial \alpha_k}{\partial x} = \alpha_k \varrho_k u_k g - p_I \Theta(p_k - p_l) - u_I \Lambda(u_k - u_l) \end{cases}$$

We assume that the drag force and pressure relaxation coefficients are given by

$$\Theta = \frac{\theta(\mathbf{U})}{\epsilon^2} \qquad \Lambda = \frac{\lambda(\mathbf{U})}{\epsilon^2}|u_1 - u_2|$$

for a small parameter ϵ . Then we have

$$p_2 - p_1 = \mathcal{O}(\epsilon^2), \qquad u_2 - u_1 = \mathcal{O}(\epsilon)$$

Asymptotic analysis

Following the Chapman-Enskog method, assume that

$$\begin{cases} p_r = p_1 - p_2 = 0 + \epsilon p_r^1 + \mathcal{O}(\epsilon^2) \\ u_r = u_1 - u_2 = 0 + \epsilon u_r^1 + \mathcal{O}(\epsilon^2) \end{cases}$$

and set

$$\begin{cases} \rho = \alpha_1\rho_1 + \alpha_2\rho_2 \\ \rho u = \alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2 \\ \rho e = \alpha_1\rho_1 e_1 + \alpha_2\rho_2 e_2 \\ \rho Y = \alpha_2\rho_2 \end{cases}$$

Theorem. A first-order approximation w.r.t. ϵ of the 7-equation model is given by the following differential drift-flux model

$$\begin{cases} \partial_t\rho + \partial_x\rho u = 0 \\ \partial_t\rho Y + \partial_x(\rho Y u + \rho Y(1-Y)u_r) = 0 \\ \partial_t\rho u + \partial_x(\rho u^2 + p + \rho Y(1-Y)u_r^2) = \rho g \\ \partial_t\rho e + \partial_x(\rho e u + p u + \rho Y(1-Y)u_r^2 u) = \rho g u \end{cases}$$

with u_r given by the (Darcy-like) differential closure relation

$$|u_r|u_r = \frac{\rho Y(\rho - \rho Y)}{\Lambda}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\frac{\partial_x p}{\rho}$$

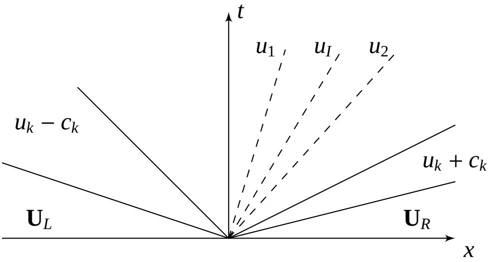
See *Ambroso-Chalons-Coquel-Galié-Godlewski-Raviart-Seguin, CMS 2008*

Mathematical properties

Eigenvalues of the Jacobian matrix $\mathbf{F}'(\mathbf{U}) + \mathbf{B}(\mathbf{U})$ are **always real** and given by

$$u_I \quad u_k \quad u_k \pm c_k \qquad k = 1, 2$$

where c_k is the sound speed of phase k



- Riemann solutions are not known and difficult to calculate/approximate
- Pressure laws may be strongly non linear, even tabulated
- Resonance occurs if $u_I = u_k \pm c_k$
- The model is not conservative...

OBJECTIVES

Note that :

- Time-step CFL restrictions are naturally based on acoustic waves
  (that are not predominant here, too bad !)

$$\max_{k,\mathbf{u}}(|u_k \pm c_k|, |u_k|, |u_I|)\,\frac{\Delta t}{\Delta x} \leq \frac{1}{2}$$

- Flows are *subsonic* and/or with *low Mach number* in nuclear reactors

Our objective is to propose a **numerical scheme** :

- able to deal with any equation of state and any choice (u_I, p_I)
- stable under a more adapted time-step CFL restriction based on transport waves
  (that are predominant here so that accuracy is required)

$$\max_{k,\mathbf{u}}(|u_k|, |u_I|)\,\frac{\Delta t}{\Delta x} \leq \frac{1}{2}$$

- and asymptotic-preserving

Asymptotic analysis (proof)

$\longrightarrow$ Using $t = s/\epsilon$ we first have

$$\begin{cases} \epsilon\partial_s\varrho + \partial_x\varrho u = 0, \\ \epsilon\partial_s\varrho u + \partial_x(\varrho u^2 + p) = \varrho g - \varrho\dfrac{\alpha}{\epsilon}u, \\ \epsilon\partial_s(\varrho E) + \partial_x(\varrho E u + pu) = \varrho u g - \varrho\dfrac{\alpha}{\epsilon}u^2 \end{cases}$$

$\longrightarrow$ Multiplying the second equation by ϵ and letting ϵ go to 0 gives

$$u_0 = 0$$

$\longrightarrow$ Then inserting $u = \epsilon u_1 + \mathcal{O}(\epsilon^2)$ in the first equation, dividing by ϵ and letting ϵ go to 0 gives

$$\partial_s\varrho + \partial_x\varrho u_1 = 0$$

$\longrightarrow$ Then inserting $u = \epsilon u_1 + \mathcal{O}(\epsilon^2)$ in the second equation and letting ϵ go to 0 gives

$$\partial_x p = \varrho g - \varrho\alpha u_1$$

$\longrightarrow$ At last inserting $u = \epsilon u_1 + \mathcal{O}(\epsilon^2)$ in the third equation, dividing by ϵ and letting ϵ go to 0 gives

$$\partial_s(\varrho e) + \partial_x(\varrho e u_1 + p u_1) = \varrho u_1 g - \varrho\alpha u_1^2$$

which concludes the proof. Back