Rigorous Foundations of the Brockett-Wegner Flow for Operators

Volker Bach   Jean-Bernard Bru

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Abstract

The Brockett-Wegner diagonalizing flow $\dot{H}_t = \big[H_t, [H_t, A]\big]$ is studied. Global existence and uniqueness of solutions of this evolution equation is proved on the space $\mathcal{B}[\mathscr{H}]$ of bounded operators on a complex Hilbert space $\mathscr{H}$. Local existence is proved for certain unbounded initial operators H_0. Furthermore, if H_0, A are Hilbert-Schmidt operators, it is demonstrated that H_t strongly converges to a diagonal operator H_∞ which is unitarily equivalent to H_0.

Keywords : Flow equations for operators, Brockett-Wegner flow, Double bracket flow, Evolution equations.

I Introduction

Almost two decades ago, Brockett [1] proposed a method to diagonalize self-adjoint complex matrices by means of a flow on the space of matrices, which we briefly outline. (He also used this method to solve linear programs, but this is of no concern here.) The notion of diagonalization he used is given in terms of the commutator $[H, A] = HA - AH$ of two self-adjoint matrices $H = H^*$ and $A = A^*$, which we assume to be complex $N \times N$ matrices, $H, A \in \mathrm{Mat}(N, \mathbb{C})$, to be specific. The matrix H is called A-*diagonal* iff $[H, A] = 0$. So, A is a reference matrix whose eigenspaces determine what is considered diagonal. If $A = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_N)$ is diagonal and possesses N different eigenvalues (of multiplicity one) then H is A-diagonal iff H is diagonal in the usual sense, i.e., $H = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$.

Brockett suggested to study the solution of the Cauchy problem

$$\forall t > 0 : \quad \dot{H}_t \;:=\; i[H_t, G_t], \qquad H_0 \;:=\; H, \tag{I.1}$$

where $G_{(\cdot)} = G^*_{(\cdot)} \in C^1\big(\mathbb{R}_0^+; \mathrm{Mat}(N, \mathbb{C})\big)$ is chosen later. He introduced a (Lyapunov) function

$$f_t \;:=\; \frac{1}{2} \mathrm{Tr}\left\{\big(H_t - A\big)^2\right\} \;\geq\; 0. \tag{I.2}$$

Note that (H_t, G_t) form a Lax pair and the solution H_t of (I.1) is unitary equivalent to H, for all $t > 0$, since G_s is self-adjoint, for all $s \geq 0$. Hence

$$f_t \;=\; \frac{1}{2} \mathrm{Tr}\left\{H_t^2 + A^2 - 2H_tA\right\} \;=\; \frac{1}{2} \mathrm{Tr}\left\{H^2 + A^2\right\} \;-\; \mathrm{Tr}\left\{H_tA\right\}. \tag{I.3}$$

Assuming the existence of a solution $H_{(\cdot)} \in C^1\big(\mathbb{R}_0^+; \mathrm{Mat}(N, \mathbb{C})\big)$ of (I.1), a differentiation yields

$$\dot{f}_t \;:=\; -\,\mathrm{Tr}\left\{\dot{H}_t\, A\right\} \;=\; -\,\mathrm{Tr}\left\{i[H_t, G_t]A\right\} \;=\; -\,\mathrm{Tr}\left\{i[A, H_t]G_t\right\}, \tag{I.4}$$

using the cyclicity of the trace. Brockett's key observation is that upon choosing

$$G_t \;=\; G_t^* \;:=\; i[A, H_t], \tag{I.5}$$

we obtain

$$\forall t \geq 0 : \qquad f_t \geq 0, \qquad -\dot{f}_t \;=\; \mathrm{Tr}\left\{G_t^2\right\} \;\geq\; 0. \tag{I.6}$$

Since f is continuously differentiable, the fundamental theorem of calculus implies that $-\dot{f} \in L^1(\mathbb{R}^+; \mathbb{R}_0^+)$ is a nonnegative, continuous, integrable function. This suggests that in the limit $t \to \infty$

(i) $[iA, H_t] \to 0$, as the square of its Hilbert-Schmidt norm equals

$$\mathrm{Tr}\left\{[iA, H_t]^2\right\} \;=\; \mathrm{Tr}\left\{G_t^2\right\} \;=\; -\dot{f}_t, \tag{I.7}$$

(ii) H_t converges to a limit matrix $H_\infty = \lim_{t\to\infty} H_t = \lim_{t\to\infty}\left\{U_{t,0} H U_{t,0}^*\right\}$ which is unitarily equivalent to H and A-diagonal. Here, $U_{t,s} \subseteq \mathrm{U}(N)$ is the cocycle (i.e., possessing the cocyclicity, or Chapman-Kolmogorov, property $U_{t,r}U_{r,s} = U_{t,s}$, for all $t \geq r \geq s \geq 0$) of unitary matrices generated by iG_t.

About the same time as Brockett, but independently, Wegner [9] used a similar idea for the diagonalization of self-adjoint operators. In the context of Eqs. (I.1)–(I.6), Wegner's choice for G_t is

$$G_t \;:=\; i\big[H_t^{\mathrm{diag}}\,,\, H_t\big]\,, \tag{I.8}$$

where H_t^{diag} denotes the diagonal part of H_t. That is, if $H_t(k,\ell) := \langle e_k | H_t e_\ell\rangle$ denotes the matrix elements of H_t w.r.t. the standard basis $\{e_n\}_{n=1}^N \subseteq \mathbb{C}^N$, then $H_t^{\mathrm{diag}}(k,\ell) := \delta_{k,\ell}\, H_t(k,k)$. Wegner's method has been successfully applied to analyze a variety of models in theoretical (mostly: condensed matter) physics, see [7]. Brockett's proposal, however, has two obvious mathematical advantages over Wegner's: It is formulated in an invariant way by means of A, fixing what is considered diagonal without referring to a particular basis. In fact, it is unclear how to extend (I.8) in case that the operators H and A have continuous spectrum and the diagonalizabilty with respect to a discrete basis is not possible. The second flaw is that Wegner's choice (I.8) for G_t is quadratic in H_t, leading to a cubicly nonlinear evolution equation for H_t, while Brockett's choice (I.5) for G_t leads to a quadratic nonlinearity. We henceforth refer to (I.1) and its solution as the *Brockett-Wegner Flow.*

We note that a few years before, Deift, Li, and Tomei [4, 5, 3] studied similar evolution equations in the context of their integrability and what is known as the Toda flow $\dot{L}_t = [B_t, L_t]$, where L_t and B_t are tridiagonal real symmetric matrices given as

$$L \;=\; \begin{pmatrix} a_1 & b_1 & & 0 \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{N-1} \\ 0 & & b_{N-1} & a_N \end{pmatrix}, \qquad B \;=\; \begin{pmatrix} 0 & b_1 & & 0 \\ -b_1 & 0 & \ddots & \\ & \ddots & \ddots & b_{N-1} \\ 0 & & -b_{N-1} & 0 \end{pmatrix}. \tag{I.9}$$

Setting $H_t := L_t$ and choosing $A = \mathrm{diag}(1, 2, \ldots, N)$, we observe that $G_t := i[A, H_t] = iB_t$, and hence the Toda flow is a special case of (I.1) and (I.5).

This is remarkable because in [4, 5, 3] the Toda flow is explicitly integrated (as an ODE), is compared to the QR-decomposition method of matrices, and variants of it are discussed, some of which become singular in finite time even in finite dimension $N < \infty$

It turns out that even for $N \times N$-matrices, the properties described under (ii) above are not obvious. Ignoring this for the time being, we remark that our outline of the Brockett-Wegner flow for $N \times N$-matrices uses the finiteness of the dimension $N^2 = \dim \mathrm{Mat}(N, \mathbb{C}) < \infty$ at various places:

(a) Inserting the definition (I.5) of G into (I.1) we obtain

$$\forall t > 0 : \quad \dot{H}_t \ := \ \big[H_t, [H_t, A]\big], \qquad H_0 \ := \ H, \tag{I.10}$$

which is a (quadratically) nonlinear differential equation, for which the global existence of solutions is not obvious, but essentially follows from the fact that the solution is a unitary conjugate of the initial value, see Sec. II.

(b) The very definition (I.2) of the Lyapunov function f employs the *trace* on $\mathrm{Mat}(N, \mathbb{C})$, and its cyclicity is crucial for the validity of Brockett's observation (I.4)-(I.6).

(c) Thus, in the infinite-dimensional context, the finiteness of f_0 would require that $H - A$ is a Hilbert-Schmidt operator.

(d) The convergence of the operators is defined in Hilbert-Schmidt norm. For many applications, this norm is not appropriate, as it is known that one can, at best, hope for strong convergence.

(f) In infinite dimensions, the unitary equivalence of all H_t and their convergence $H_t \to H_\infty$ do not necessarily imply that the limiting operator H_∞ is unitarily equivalent to $H = H_0$, because the family $U_{t,s}$ of unitary operators generated by iG_t may not converge.

(g) In fact, in infinite dimensions, the distinguished spectral types (ac, sc, pp, etc.) characterizing the spectral measure of H are unitary invariants. So, for instance, if H has continuous spectrum then it cannot be forced to become diagonal in any orthonormal basis. Then either H_t does not converge at all or it does converge but not to a diagonal operator or its limit is not unitarily equivalent to H. Either way, the flow will not diagonalize H.

The results in the present paper are the following.

- We first show that (I.10) has a unique, smooth *global* solution $H_{(\cdot)} \in C^\infty(\mathbb{R}_0^+; \mathcal{B}[\mathscr{H}])$, provided $H_0, A \in \mathcal{B}[\mathscr{H}]$ are bounded operators. There is a family $U_{(\cdot)} \in C^\infty(\mathbb{R}_0^+; \mathcal{B}[\mathscr{H}])$ of unitary operators such that all $H_t = U_t H_0 U_t^*$ are (mutually) unitarily equivalent. This is carried out in Sect. II.

- Secondly, using a Nash-Moser type of estimate, we show that (I.10) has a unique, smooth *local* solution $H_{(\cdot)} \in C^\infty([0,T]; \mathcal{B}[\mathscr{H}])$ in case of unbounded H_0 provided its iterated commutators with A define bounded operators whose norm tends to zero, as the order increases, sufficiently fast. The precise formulation of the assertion and its proof is the contents of Sect. III.

- Thirdly, under the additional assumption that $H_0, A \in \mathcal{L}^2[\mathscr{H}]$ are Hilbert-Schmidt operators, we show in Sect. IV that H_t strongly converges to a diagonal operator H_∞ which is unitarily equivalent to H_0.

We conclude this introduction with a few remarks. A typical situation encountered in quantum mechanics, say, is that both H and A are unbounded, but bounded below. In this case $H + \lambda$ and $A + \lambda$ are positive and hence bounded invertible, provided $\lambda > 0$ is sufficiently big. The A-diagonalization of H is clearly equivalent to the $(A+\lambda)^{-1}$-diagonalization of $(H+\lambda)^{-1}$. So, it seems that the diagonalization of unbounded, but semibounded operators can be traced back to the diagonalization of bounded operators. This is, however, not really the case because the differential equation for H_t changes - even if H and A are strictly positive. Namely, if

$$\partial_t\big(H_t^{-1}\big) \; = \; \big[H_t^{-1}, [H_t^{-1}, A^{-1}]\big] \, , \tag{I.11}$$

then

$$\dot{H}_t \; = \; \big[H_t, [H_t^{-1}, A^{-1}]\big] \; = \; -H_t^{-1} \big[H_t, [H_t, A^{-1}]\big] \, H_t^{-1} \, , \tag{I.12}$$

which is not identical to (I.10) - even if we ignore the change from A to A^{-1}. In fact, (I.12) is more involved than (I.10) because it additionally requires to invert H_t for the computation of its right side.

One may also consider passing from H and A to $e^{-\beta H}$ and $e^{-\beta A}$ in the case of semibounded H and A, but this suffers from the same flaw: computing the right side of the differential equation requires the explicit computation of $e^{-\beta H}$ and $e^{-\beta A}$.

Another observation is that without specification of G_t, Eq. (I.1) is a linear, non-autonomous evolution equation. The theory of these is known to be much more difficult than the one of autonomous evolution equations. The latter are fully characterized by the Hille-Yoshida theorem which essentially says that the solutions of autonomous evolution equations are precisely strongly continuous hyperbolic semigroups generated by densely defined operators fulfilling Kato's (quasi-)stability condition. In contrast, in the non-autonomous case, only sufficient conditions, basically extending the autonomous case, are known. The necessity of these conditions remains unclear until today. This subject has a long history, and important contributions have been made by Kato, Yosida, Tanabe, Kisynski, Hackman, Kobayasi, Ishii, Goldstein, Acquistapace, Terreni, Nickel, Schnaubelt, Caps, Tanaka, and many more; see for instance [6, 2] and references therein. From this angle, the problem treated in Sect. III seems trivial: the $(G_t)_{t\geq 0}$ is a family of bounded self-adjoint operators. The preservation of the domain of H_t under the flow generated by G_t, however, is difficult to control because G_t itself depends on H_t. This is the achievement of Sect. III.

II Global Existence on Bounded Operators

Our first task is to show that the Cauchy problem stated in (I.10), that is

$$\forall t > 0 : \quad \dot{H}_t \ := \ \big[H_t, [H_t, A]\big], \qquad H_0 \ := \ H, \tag{II.13}$$

possesses a unique global solution $(H_t)_{t\geq 0}$ which is smooth in t. We formulate this result in the following theorem

Theorem 1 *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach subalgebra of the Banach algebra $\mathcal{B}[\mathscr{H}] \supset \mathcal{A}$ of bounded operators on a separable Hilbert space $\mathscr{H}$ such that $\|\cdot\|_{\mathcal{A}}$ is a unitarily invariant norm. Suppose that $H_0 = H_0^*, A = A^* \in \mathcal{A}$ are two self-adjoint operators such that $A \geq 0$. Then Eq.* (II.13) *has a unique solution $H_{(\cdot)} \in C^\infty(\mathbb{R}_0^+; \mathcal{A})$, and H_t is unitarily equivalent to H_0, for all $t > 0$.*

Proof: The assumption on $\mathcal{A}$ to be a Banach algebra with unitarily invariant norm means that $\|AB\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}} \cdot \|B\|_{\mathcal{A}}$ and $\|UAU^*\|_{\mathcal{A}} = \|A\|_{\mathcal{A}}$, for all $A, B \in \mathcal{A}$ and all unitary operators $U \in \mathcal{B}[\mathscr{H}]$.

The first part of the argument employs the standard contraction mapping principle in connection to the Picard iteration. We fix $A \in \mathcal{A}$ and define $F : \mathcal{A} \to \mathcal{A}$ by

$$F(H) \ := \ \big[H, [H, A]\big]. \tag{II.14}$$

We observe for any $H, \widetilde{H} \in \mathcal{A}$ that

$$F(H) - F(\widetilde{H}) \ = \ \big[H - \widetilde{H}, [H, A]\big] \ + \ \big[\widetilde{H}, [H - \widetilde{H}, A]\big] \tag{II.15}$$

which implies that F is locally Lipschitz continuous and that $t \mapsto F(H_t)$ is differentiable in norm if $t \mapsto H_t$ is, namely

$$\|F(H) - F(\widetilde{H})\|_{\mathcal{A}} \ \leq \ 4\, \|A\|_{\mathcal{A}} \big(\|H\|_{\mathcal{A}} + \|\widetilde{H}\|_{\mathcal{A}}\big)\, \|H - \widetilde{H}\|_{\mathcal{A}}, \tag{II.16}$$

and

$$\frac{dF(H_t)}{dt} \ = \ \big[\dot{H}_t, [H_t, A]\big] \ + \ \big[H_t, [\dot{H}_t, A]\big]. \tag{II.17}$$

Denoting by $Y_T := C\big([0, T]; \mathcal{A}\big)$ the Banach space w. r. t. the norm $\|M_{(\cdot)}\| := \max_{0\leq t\leq T} \|M_t\|_{\mathcal{A}}$, as usual, we conclude that $\mathcal{F}_T : C\big([0, T]; \mathcal{A}\big) \to C\big([0, T]; \mathcal{A}\big)$, defined by

$$\big(\mathcal{F}_T(M_{(\cdot)})\big)_t \ := \ H_0 \ + \ \int_0^t F(M_s)\, ds \tag{II.18}$$

defines a contraction on

$$D_T \ := \ \Big\{ M_{(\cdot)} \in Y_T \ \Big| \ \max_{0\leq t\leq T} \|M_t\|_{\mathcal{A}} \leq (16\, \|A\|_{\mathcal{A}}\, T)^{-1} \Big\}. \tag{II.19}$$

Thus $M_{(\cdot)} = \mathcal{F}_T(M_{(\cdot)})$ has a unique smooth solution which we denote $H_{(\cdot)} \in C^\infty\big([0,T];\mathcal{A}\big)$ provided

$$T \;:=\; \frac{1}{32\,\|A\|_{\mathcal{A}}\,\|H_0\|_{\mathcal{A}}}. \tag{II.20}$$

The key observation is that the self-adjointness of $G_t = i[A, H_t] \in \mathcal{A}$ implies that the unique solution $U_{(\cdot)} \in C^\infty\big([0,T];\mathcal{A}\big)$ of the linear, non-autonomous evolution equation

$$\forall t > 0: \quad \dot{U}_t \;:=\; -iG_t\,U_t, \qquad U_0 \;:=\; \mathbf{1}, \tag{II.21}$$

is unitary. Moreover, $H_t = U_t H_0 U_t^*$ and in particular

$$\|H_T\|_{\mathcal{A}} \;=\; \|U_T\,H_0\,U_T^*\|_{\mathcal{A}} \;=\; \|H_0\|_{\mathcal{A}}. \tag{II.22}$$

We can therefore repeat the argument starting from (II.18), replacing H_0 by H_T. Clearly, an iteration of this procedure yields the desired global solution.

□

III Local Existence on Unbounded Operators

This section is devoted to proving local existence and uniqueness of solutions of the Brockett-Wegner flow for unbounded operators H_0 which, together with its iterated commutators, are bounded relative to A, where we assume w. l. o. g. that $A \geq 1$. Instead of (I.10), we study

$$\forall t > 0 : \quad \dot{H}_t \ := \ \big[H_t, [H_t, A^{-1}]\big], \qquad H_0 \ := \ H, \tag{III.23}$$

in this section, i.e., we replace A by A^{-1}. The proof of our result in Theorem 2 below was inspired by a similar result of Caps [2] for quasilinear evolution equations on scales of Banach spaces, using estimates of Nash-Moser type, see [8].

To formulate our result, we introduce some notation. We denote by $X := \mathcal{B}[\mathscr{H}]$ the space of bounded operators on a complex, separable Hilbert space $\mathscr{H}$. We assume that $A \geq \mathbf{1}$ is a positive operator and H a closed operator with dense common domain $\mathscr{D} \subseteq \mathscr{H}$ which is itself a Hilbert space w.r.t. $\langle x|y\rangle_{\mathscr{D}} := \langle Ax|Ay\rangle_{\mathscr{H}}$. We denote by $Y := \mathcal{B}[\mathscr{D}]$ the bounded operators on $\mathscr{D}$ and note the natural identification $\|M\|_Y = \|AMA^{-1}\|_X$.

We further assume that $HA^{-1} \in X$ and $R_n(H) \in X \cap Y$, for all $n \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$, where $R_n(H)$ is recursively defined by $R_0(H) := A^{-1}$ and

$$\forall\, n \in \mathbb{N} : \qquad R_n(H) \ := \ [H, R_{n-1}(H)] \ = \ \mathrm{Ad}_H^n[A^{-1}], \tag{III.24}$$

with $[A, B] := AB - BA$, as usual. More precisely, for a given $n \in \mathbb{N}$ and $R_{n-1}(H) \in X \cap Y$, we first define $R_n(H) \in \mathcal{B}[\mathscr{D}; \mathscr{H}]$ by

$$R_n(H)\, A^{-1} \ := \ \big(HA^{-1}\big)\big(AR_{n-1}(H)A^{-1}\big) \ - \ R_{n-1}(H)\big(HA^{-1}\big) \tag{III.25}$$

and assume that $R_n(H)$ extends to an operator bounded both on $\mathscr{D}$ and on $\mathscr{H}$, i.e., $R_n(H) \in X \cap Y$.

Theorem 2 *Suppose that $H_0 = H_0^*$ and $A = A^* \geq \mathbf{1}$ are two self-adjoint operators on $\mathscr{H} \supseteq \mathscr{D}$ and that $R_n(H_0) : \mathscr{D} \to \mathscr{H}$ extend to bounded operators $R_n(H_0) \in X \cap Y$, for all $n \in \mathbb{N}$, such that $\|H_0 A^{-1}\|_X < \infty$ and*

$$\sum_{n=0}^{\infty} \frac{e^{\rho n}}{n!} \big(\|R_n(H_0)\|_X + \|R_n(H_0)\|_Y\big) \ \leq \ e^{\eta}\,, \tag{III.26}$$

for some $\rho, \eta \in \mathbb{R}$. Then the following assertions hold true.

(i) *Eq.* (III.23) *has a self-adjoint solution* $H_{(\cdot)} \in C^\infty\big([0,T_*);\mathcal{B}[\mathscr{D},\mathscr{H}]\big)$, *where* $T_* := \frac{1}{8}e^{\rho-\eta-1}$, *and there exists a smooth family* $U_{(\cdot)} \in C^\infty\big([0,T_*); X\cap Y\big)$ *of unitary tranformations preserving the domain* $\mathscr{D}$ *of* H_t *such that* $H_t = U_t H_0 U_t^*$ *is unitarily equivalent to* H_0, *for all* $t \in [0,T_*)$.

(ii) *For all* $T < T_*$, *the solution* $H_{(\cdot)} \in C^\infty\big([0,T_*);\mathcal{B}[\mathscr{D},\mathscr{H}]\big)$ *obeys*

$$\sum_{n=0}^\infty \frac{e^{\rho n}}{n!} \max_{t\in[0,T]} \big\{ e^{-nt/T_*} \big(\|R_n(H_t)\|_X + \|R_n(H_t)\|_Y \big) \big\} \ \le\ 2\, e^\eta , \quad \text{(III.27)}$$

and $H_{(\cdot)}$ *is the only solution with this property.*

We break up the proof of Theorem 2 into several steps.

Our first task is to map (III.23) to an auxiliary problem for any solution $H_{(\cdot)} \in C^1(\mathbb{R}_0^+;X)$, for which the operators $R_n(H_t) \in X\cap Y$. Lemma 3 below derives a system of evolution equations on $\underline{R}(t)$ from (III.23), where $\underline{R}(t) := \big(R_n(H_t)\big)_{n=0}^\infty$.

Lemma 3 *Let* $H_{(\cdot)} \in C^1\big(\mathbb{R}_0^+;\mathcal{B}[\mathscr{D};\mathscr{H}]\big)$ *be a solution of the Cauchy problem* (III.23) *such that* $R_n := R_n(H_{(\cdot)}) \in C^1\big(\mathbb{R}_0^+; X\cap Y\big)$, *for all* $n\in\mathbb{N}_0$. *Then*

$$\dot R_n(t) \ = \ F_n[\underline{R}(t)] \ := \ \sum_{\nu=1}^n \binom{n}{\nu} [R_{\nu+1}(t)\,,\, R_{n-\nu}(t)]\,, \quad \text{(III.28)}$$

for all $n\in\mathbb{N}_0$.

Proof: We use an induction in $n\ge 0$. First, $R_0(t) = A^{-1}$ is independent of t, and so we have 0 on both sides of (III.28) for $n=0$.

Now suppose that (III.28) holds true for $n\ge 0$. To understand the structure of the next computation, assume temporarily that $H_t\in X$ is bounded. Then $R_{n+1}(H_t) = [H_t, R_n(H_t)]$, Leibniz' rule, and $\dot H_t = R_2(t)$ would yield

$$\begin{aligned}\dot R_{n+1}(t) \ &= \ [\dot H_t\,,\, R_n(t)] \ + \ [H_t\,,\, \dot R_n(t)] \quad \text{(III.29)}\\ &= \ [R_2(t)\,,\, R_n(t)] \ + \ \sum_{\nu=1}^n \binom{n}{\nu} \big[H_t\,,\, [R_{\nu+1}(t)\,,\, R_{n-\nu}(t)]\big]\,,\end{aligned}$$

where we use that all operators are bounded. By Jacobi's identity, we have that

$$\begin{aligned}&\big[H_t\,,\, [R_{\nu+1}(t)\,,\, R_{n-\nu}(t)]\big]\\ &= \ \big[[H_t\,,\, R_{\nu+1}(t)]\,,\, R_{n-\nu}(t)\big] \ + \ \big[R_{\nu+1}(t)\,,\, [H_t\,,\, R_{n-\nu}(t)]\big]\\ &= \ [R_{\nu+2}(t)\,,\, R_{n-\nu}(t)] \ + \ [R_{\nu+1}(t)\,,\, R_{n-\nu+1}(t)]. \quad \text{(III.30)}\end{aligned}$$

Since H_t is not bounded but only bounded relative to A, carrying out (III.29) and (III.30) require an additional argument, in general. Denoting $H := H_t$, $H^{(A)} := H_t A^{-1}$, $R_k := R_k(t)$, and $R_k^{(A)} := A R_k(t) A^{-1}$, we observe that according to (III.25), we have $R_{n+1}A^{-1} := H^{(A)} R_n^{(A)} - R_n H^{(A)}$ and thus

$$\dot{R}_{n+1}A^{-1} \;=\; \dot{H}^{(A)}\, R_n^{(A)} \;-\; R_n\, \dot{H}^{(A)} \;+\; H^{(A)}\, \dot{R}_n^{(A)} \;-\; \dot{R}_n\, H^{(A)} \tag{III.31}$$

$$= \;\; [R_2\,,\, R_n]A^{-1} \;+\; \sum_{\nu=1}^{n} \binom{n}{\nu} \big(H^{(A)}\, [R_{\nu+1}^{(A)}\,,\, R_{n-\nu}^{(A)}] \;-\; [R_{\nu+1}\,,\, R_{n-\nu}]\, H^{(A)} \big)\,.$$

Similar to Jacobi's identity, we obtain

$$\begin{aligned} & H^{(A)}\, [R_{\nu+1}^{(A)}\,,\, R_{n-\nu}^{(A)}] \;-\; [R_{\nu+1}\,,\, R_{n-\nu}]\, H^{(A)} \qquad\qquad \text{(III.32)} \\ & = \;\; H^{(A)}\, R_{\nu+1}^{(A)}\, R_{n-\nu}^{(A)} \;-\; H^{(A)}\, R_{n-\nu}^{(A)}\, R_{\nu+1}^{(A)} \\ & \qquad + R_{n-\nu}\, R_{\nu+1}\, H^{(A)} \;-\; R_{\nu+1}\, R_{n-\nu}\, H^{(A)} \\ & = \;\; \big(H^{(A)} R_{\nu+1}^{(A)} - R_{\nu+1}\, H^{(A)} \big) R_{n-\nu}^{(A)} \;-\; \big(H^{(A)} R_{n-\nu}^{(A)} - R_{n-\nu}\, H^{(A)} \big) R_{\nu+1}^{(A)} \\ & \qquad + R_{\nu+1} \big(H^{(A)} R_{n-\nu}^{(A)} - R_{n-\nu}\, H^{(A)} \big) \;-\; R_{n-\nu} \big(H^{(A)} R_{\nu+1}^{(A)} - R_{\nu+1}\, H^{(A)} \big) \\ & = \;\; [R_{\nu+2}\,,\, R_{n-\nu}]\, A^{-1} \;+\; [R_{\nu+1}\,,\, R_{n-\nu+1}]\, A^{-1} \qquad\qquad \text{(III.33)} \end{aligned}$$

and thus

$$\dot{R}_{n+1} \;=\; [R_2\,,\, R_n] \;+\; \sum_{\nu=1}^{n} \binom{n}{\nu} \Big([R_{\nu+2}\,,\, R_{n-\nu}] \;+\; [R_{\nu+1}\,,\, R_{n-\nu+1}] \Big) \tag{III.34}$$

holds on $\mathscr{D}$ and by continuity hence also on $\mathscr{H}$. Inserting this into (III.29), we obtain

$$\begin{aligned} \dot{R}_{n+1}(t) \;\; & = \;\; [R_2(t)\,,\, R_n(t)] \;+\; \sum_{\nu=1}^{n} \binom{n}{\nu} [R_{\nu+2}(t)\,,\, R_{n-\nu}(t)] \\ & \qquad + \sum_{\nu=1}^{n} \binom{n}{\nu} [R_{\nu+1}(t)\,,\, R_{n+1-\nu}(t)] \qquad\qquad \text{(III.35)} \\ & = \;\; \sum_{\nu=0}^{n} \binom{n}{\nu} [R_{\nu+2}(t)\,,\, R_{n-\nu}(t)] \;+\; \sum_{\nu=1}^{n} \binom{n}{\nu} [R_{\nu+1}(t)\,,\, R_{n+1-\nu}(t)] \\ & = \;\; \sum_{\nu=1}^{n+1} \binom{n}{\nu-1} [R_{\nu+1}(t)\,,\, R_{n+1-\nu}(t)] \;+\; \sum_{\nu=1}^{n} \binom{n}{\nu} [R_{\nu+1}(t)\,,\, R_{n+1-\nu}(t)] \\ & = \;\; \sum_{\nu=1}^{n+1} \binom{n+1}{\nu} [R_{\nu+1}(t)\,,\, R_{n+1-\nu}(t)], \end{aligned}$$

which completes the induction step and hence the proof of (III.28). □

We can rewrite (III.28) as the integral equation

$$\underline{r}(t) \;=\; \mathcal{F}[\underline{r}](t) \;:=\; \underline{r}^{(0)} \;+\; \int_0^t F[\underline{r}(\tau)]\, d\tau \tag{III.36}$$

with initial value $\underline{r}^{(0)} = \underline{R}(0)$ and $F(\underline{r}) := \big(F_n(\underline{r})\big)_{n=0}^{\infty}$. Next we show that any solution $\underline{R}$ of (III.36) is continuous at $t \searrow 0$ and of moderate growth for small t, provided the initial data $\underline{R}(0)$ is sufficiently regular.

For the precise formulation of this statement we fix a triple $\theta := (\alpha, T, \rho)$ consisting of a positive parameter $\alpha > 0$, a time $T > 0$, and a real number $\rho \in \mathbb{R}$. Furthermore $(X \cap Y)^{\mathbb{N}_0}$ denotes the space of sequences $(x_0, x_1, x_2, \ldots)$ with $x_j \in X \cap Y$. Then we introduce the Banach space

$$\mathcal{R}_\theta \;:=\; C\big([0,T]\,;\; (X \cap Y)^{\mathbb{N}_0}\big)\,, \tag{III.37}$$

which we equip with the norm

$$\|\underline{r}\|_\theta \;:=\; \sum_{n=0}^{\infty} \frac{e^{\rho n}}{n!} \max_{t \in [0,T]} \big\{ e^{-\alpha n t} \|r_n(t)\|_{X \cap Y} \big\}\,, \tag{III.38}$$

where $\underline{r} = \big(r_n(\cdot)\big)_{n=0}^{\infty} \in \mathcal{R}_\theta$ and $\|a\|_{X \cap Y} := \|a\|_X + \|a\|_Y$. Note that this norm is again submultiplicative because the norms on X and Y are, i.e.,

$$\begin{aligned} \|ab\|_{X\cap Y} \;&=\; \|ab\|_X + \|ab\|_Y \;\le\; \|a\|_X \|b\|_X + \|a\|_Y \|b\|_Y \\ &\le\; \big(\|a\|_X + \|a\|_Y\big)\big(\|b\|_X + \|b\|_Y\big) \;=\; \|a\|_{X\cap Y} \|b\|_{X \cap Y}\,. \end{aligned} \tag{III.39}$$

Now we are in position to show that $\mathcal{F}$ is locally Lipschitz-continuous on $\mathcal{R}_\theta$.

Lemma 4 *Let $\theta = (\alpha, T, \rho)$ with $\alpha > 0$, $T < \infty$, $\rho \in \mathbb{R}$, and $\underline{r}^{(0)} \in \mathcal{R}_\theta$. Then, for all $\underline{r}, \underline{\hat{r}} \in \mathcal{R}_\theta$, we have*

$$\big\|\mathcal{F}[\,\underline{r}\,] - \mathcal{F}[\,\underline{\hat{r}}\,]\big\|_\theta \;\le\; \frac{2\, e^{\alpha T - \rho}}{\alpha} \Big(\big\|\underline{r}\big\|_\theta + \big\|\underline{\hat{r}}\big\|_\theta \Big) \, \big\|\underline{r} - \underline{\hat{r}}\big\|_\theta\,. \tag{III.40}$$

Proof: We first remark that

$$\begin{aligned} F_n[\underline{r}] \,-\, F_n[\underline{\hat{r}}] \;&=\; \sum_{\nu=1}^{n} \binom{n}{\nu} \Big\{ [r_{\nu+1}\,,\, r_{n-\nu}] \,-\, [\hat{r}_{\nu+1}\,,\, \hat{r}_{n-\nu}] \Big\} \\ &=\; \sum_{\nu=1}^{n} \binom{n}{\nu} \Big([r_{\nu+1} - \hat{r}_{\nu+1}\,,\, r_{n-\nu}] \,+\, [\hat{r}_{\nu+1}\,,\, r_{n-\nu} - \hat{r}_{n-\nu}] \Big)\,, \end{aligned} \tag{III.41}$$

and thus

$$\begin{aligned}
&\big\| \mathcal{F}[\underline{r}] \, - \, \mathcal{F}[\underline{\hat{r}}] \big\|_\theta \qquad \text{(III.42)} \\
&\le \sum_{n=0}^{\infty} \frac{e^{\rho n}}{n!} \max_{t \in [0,T]} \bigg\| e^{-\alpha n t} \sum_{\nu=1}^{n} \binom{n}{\nu} \int_0^t \Big([r_{\nu+1}(\tau) - \hat{r}_{\nu+1}(\tau) \, , \, r_{n-\nu}(\tau)] \\
&\qquad\qquad + [\hat{r}_{\nu+1}(\tau) \, , \, r_{n-\nu}(\tau) - \hat{r}_{n-\nu}(\tau)] \Big) \, d\tau \bigg\|_{X \cap Y} \\
&\le \sum_{n=0}^{\infty} \sum_{\nu=1}^{n} \max_{t \in [0,T]} \bigg\{ \frac{2 \, e^{\rho n} \, e^{-\alpha n t}}{\nu! \, (n-\nu)!} \int_0^t \Big(\big\| r_{\nu+1}(\tau) - \hat{r}_{\nu+1}(\tau) \big\|_{X \cap Y} \, \big\| r_{n-\nu}(\tau) \big\|_{X \cap Y} \\
&\qquad\qquad + \big\| \hat{r}_{\nu+1}(\tau) \big\|_{X \cap Y} \big\| r_{n-\nu}(\tau) - \hat{r}_{n-\nu}(\tau) \big\|_{X \cap Y} \Big) \, d\tau \\
&\le \sum_{n=0}^{\infty} \sum_{\nu=1}^{n} \frac{2(\nu+1)}{e^{\rho}} \max_{t \in [0,T]} \bigg\{ e^{-\alpha n t} \int_0^t e^{\alpha(n+1)\tau} \, d\tau \bigg\} \Big(d_{\nu+1} c_{n-\nu} \, + \, \hat{c}_{\nu+1} d_{n-\nu} \Big) \, ,
\end{aligned}$$

where

$$c_n \; := \; \frac{e^{\rho n}}{n!} \max_{t \in [0,T]} \big\{ e^{-\alpha n t} \| r_n(t) \|_{X \cap Y} \big\} \, , \qquad \text{(III.43)}$$

$$\hat{c}_n \; := \; \frac{e^{\rho n}}{n!} \max_{t \in [0,T]} \big\{ e^{-\alpha n t} \| \hat{r}_n(t) \|_{X \cap Y} \big\} \, , \qquad \text{(III.44)}$$

$$d_n \; := \; \frac{e^{\rho n}}{n!} \max_{t \in [0,T]} \big\{ e^{-\alpha n t} \| r_n(t) - \hat{r}_n(t) \|_{X \cap Y} \big\} \, . \qquad \text{(III.45)}$$

We observe that

$$\frac{\nu+1}{e^{\rho}} \max_{t \in [0,T]} \bigg\{ e^{-\alpha n t} \int_0^t e^{\alpha(n+1)\tau} \, d\tau \bigg\} \; \le \; \frac{\nu+1}{\alpha \, (n+1)} \max_{t \in [0,T]} \big\{ e^{\alpha t - \rho} \big\} \; \le \; \frac{e^{\alpha T - \rho}}{\alpha} \, . \qquad \text{(III.46)}$$

Inserting this into (III.42) and using $\sum_{n=0}^{\infty} c_n = \|\underline{r}\|_\theta$, $\sum_{n=0}^{\infty} \hat{c}_n = \|\underline{\hat{r}}\|_\theta$, and $\sum_{n=0}^{\infty} d_n = \|\underline{r} - \underline{\hat{r}}\|_\theta$, we arrive at (III.40). □

The following Lemma 5 asserts that $\mathcal{F}$ maps small balls in $\mathcal{R}_\theta$ into itself. The proof is omitted, as it is very similar to the one for Lemma 4.

Lemma 5 *Let $\theta = (\alpha, T, \rho)$ with $\alpha > 0$, $T < \infty$, $\rho \in \mathbb{R}$, and $\underline{r}^{(0)} \in \mathcal{R}_\theta$. Then, for all $\underline{r} \in \mathcal{R}_\theta$, we have*

$$\big\| \mathcal{F}[\,\underline{r}\,] \big\|_\theta \; \le \; \big\| \underline{r}^{(0)} \big\|_\theta + \frac{2 \, e^{\alpha T - \rho}}{\alpha} \, \big\| \underline{r} \big\|_\theta^2 . \qquad \text{(III.47)}$$

Proof of Theorem 2:
(i) We first choose $\alpha := 8e^{1+\eta-\rho} = T_*^{-1}$ and $T < T_*$, so that $\alpha T < 1$ and

$$\frac{2\,e^{\alpha T-\rho}}{\alpha} \;<\; \frac{e^{1-\rho}}{4\,e^{1+\eta-\rho}} \;=\; \frac{e^{-\eta}}{4}\,. \tag{III.48}$$

We set $r_n^{(0)} := R_n(H_0)$ and $\underline{r}^{(0)} := \big(r_n^{(0)}\big)_{n=0}^{\infty}$. Identifying $\underline{r}^{(0)}$ with the corresponding constant vector $\underline{r}^{(0)}(\cdot) := \underline{r}^{(0)} \in \mathcal{R}_\theta$, we have

$$\big\|\underline{r}^{(0)}\big\|_\theta \;\le\; e^{\eta}\,, \tag{III.49}$$

with $\theta = (\alpha, T, \rho)$. In particular,

$$\underline{r}^{(0)} \;\in\; B \;:=\; \Big\{\underline{r} \in \mathcal{R}_\theta \;\Big|\; \big\|\underline{r}\big\|_\theta \;\le\; 2\,e^{\eta}\Big\}\,. \tag{III.50}$$

Moreover, for $\underline{r} \in B$, we observe that due to Lemma 5, (III.48), and (III.49), we have

$$\big\|\mathcal{F}[\,\underline{r}\,]\big\|_\theta \;\le\; \big\|\underline{r}^{(0)}\big\|_\theta + \frac{e^{-\eta}}{4}\,\big\|\underline{r}\big\|_\theta^2 \;\le\; 2\,e^{\eta} \tag{III.51}$$

and thus $\mathcal{F}$ leaves B invariant, i.e.,

$$\mathcal{F}[B] \;\subseteq\; B\,. \tag{III.52}$$

Furthermore, for any $\underline{r}, \underline{\hat{r}} \in B$ with $\underline{r} \neq \underline{\hat{r}}$, we have

$$\frac{\big\|\mathcal{F}[\,\underline{r}\,] \;-\; \mathcal{F}[\,\underline{\hat{r}}\,]\big\|_\theta}{\big\|\underline{r} - \underline{\hat{r}}\big\|_\theta} \;\le\; \frac{2\,e^{\alpha T-\rho}}{\alpha}\,(2e^{\eta} + 2e^{\eta}) \;<\; 1\,, \tag{III.53}$$

and $\mathcal{F} : B \to B$ is a contraction. By the contraction mapping principle, $\mathcal{F}$ has a unique fixed point in B which we denote $\underline{S}$. Clearly, $\underline{S} =: \big(S_n(\cdot)\big)_{n=0}^{\infty}$ is the unique $(X \cap Y)^{\mathbb{N}_0}$-valued smooth solution of (III.28) with initial value $\big(S_n(0)\big)_{n=0}^{\infty} = \big(R_n(H_0)\big)_{n=0}^{\infty}$. In particular,

$$iS_1 \in C^\infty\big([0,T]; X\cap Y\big)\,, \quad iS_1(t) = -iS_1^*(t) \in X\,, \quad \|S_1(t)\|_{X\cap Y} \le 2\,e^{\eta-\rho+\alpha t}\,, \tag{III.54}$$

for all $t \in [0,T]$, defines a smooth family of bounded self-adjoint operators, and thus

$$\dot{U}_t \;=\; -S_1(t)\,U_t\,, \quad U_0 = \mathbf{1} \tag{III.55}$$

defines a smooth family $U_{(\cdot)} \in C^\infty\big([0,T]\;;\; X \cap Y\big)$ of unitary operators on $\mathscr{H}$. Equivalently to (III.55), the family $U_{(\cdot)}$ is determined by $\dot{U}_t^* = U_t^* S_1(t)$ and $U_0^* = \mathbf{1}$. Hence, we have the norm bound

$$\|U_t^*\|_Y \;\le\; \exp\left[\int_0^t \|S_1(\tau)\|_Y\, d\tau\right] \;\le\; \exp\left[\frac{2e^{\eta-\rho+\alpha t}}{\alpha}\right] \;\le\; \exp\left[\frac{e^{-\alpha(T-t)}}{4}\right]. \tag{III.56}$$

Setting

$$H_t \;:=\; U_t\, H_0\, U_t^* \,, \tag{III.57}$$

we observe that $H_{(\cdot)} \in C^\infty\big([0,T]; \mathcal{B}[\mathscr{D}, \mathscr{H}]\big)$ is a smooth family of self-adjoint operators defined on $\mathscr{D}$ obeying

$$\|H_t A^{-1}\|_X \;\le\; \|H_0 A^{-1}\|_X \cdot \|U_t\|_Y \;\le\; 2\, \|H_0 A^{-1}\|_X \,, \tag{III.58}$$

for all $t \in [0,T]$. (Here we use $e^{s/4} \le 2$ for $s \le 1$.) Moreover, $H_{(\cdot)}$ solves the Cauchy Problem

$$\dot{H}_t \;=\; [H_t \,,\, S_1(t)] \,, \quad H_0 = H \,. \tag{III.59}$$

We introduce the family $\underline{Q} := (Q_n)_{n\in\mathbb{N}} \in C^\infty\big([0,T]; (X\cap Y)^{\mathbb{N}_0}\big)$ by

$$\forall\, n \in \mathbb{N}_0,\ t\in[0,T] \,:\quad Q_n(t) \;=\; [H_t \,,\, S_n(t)] \;-\; S_{n+1}(t) \,. \tag{III.60}$$

Note that the initial value vanishes, $Q_n(0) = 0$, for all $n \in \mathbb{N}$. We observe that (suppressing the time parameter) by Leibniz' rule and Jacobi's identity

$$\begin{aligned} \partial_t [H, S_n] \;&=\; [\dot{H}, S_n] \;+\; [H, \dot{S}_n] \qquad\qquad \text{(III.61)} \\ =\;& \big[[H,S_1],\, S_n\big] \;+\; \sum_{\nu=1}^n \binom{n}{\nu} \big[H,\, [S_{\nu+1}, S_{n-\nu}]\big] \\ =\;& \big[[H,S_1],\, S_n\big] \;+\; \sum_{\nu=1}^n \binom{n}{\nu} \Big\{ \big[[H, S_{\nu+1}], S_{n-\nu}\big] \;+\; \big[S_{\nu+1}, [H, S_{n-\nu}]\big] \Big\} \,, \end{aligned}$$

while, as in (III.34)

$$\dot{S}_{n+1} \;=\; [S_2 \,,\, S_n] \;+\; \sum_{\nu=1}^n \binom{n}{\nu} \Big([S_{\nu+2} \,,\, S_{n-\nu}] \;+\; [S_{\nu+1} \,,\, S_{n-\nu+1}] \Big) \,. \tag{III.62}$$

Consequently,

$$\dot{Q}_n \;=\; [Q_1 \,,\, S_n] \;+\; \sum_{\nu=1}^n \binom{n}{\nu} \Big([Q_{\nu+1} \,,\, S_{n-\nu}] \;+\; [S_{\nu+1} \,,\, Q_{n-\nu}] \Big) \,, \tag{III.63}$$

which in turn implies that

$$\begin{aligned} &\frac{\|\dot{Q}_n\|_{X\cap Y}}{n!} \;\le \qquad\qquad \text{(III.64)} \\ &2 \sum_{\nu=0}^n (\nu+1) \bigg\{ \frac{\|Q_{\nu+1}\|_{X\cap Y}}{(\nu+1)!} \, \frac{\|S_{n-\nu}\|_{X\cap Y}}{(n-\nu)!} \;+\; \frac{\|S_{\nu+1}\|_{X\cap Y}}{(\nu+1)!} \, \frac{\|Q_{n-\nu}\|_{X\cap Y}}{(n-\nu)!} \bigg\} \,. \end{aligned}$$

Now, an estimation as in (III.42), taking $Q_n(0) = 0$ and $\|\underline{S}\|_\theta \leq 2e^\eta$ into account, yields

$$\|\underline{Q}\|_\theta \;\leq\; 4\,\frac{e^{\alpha T-\rho}}{\alpha}\,\|\underline{S}\|_\theta\,\|\underline{Q}\|_\theta \;\leq\; \frac{e^{\alpha T-\rho}}{\alpha}\,\frac{8}{e^{-\eta}}\,\|\underline{Q}\|_\theta\,. \tag{III.65}$$

Since $\frac{e^{\alpha T-\rho}}{\alpha} < \frac{e^{-\eta}}{8}$, this estimate implies that $\|\underline{Q}\|_\theta = 0$, i.e.,

$$\forall\, n \in \mathbb{N}_0,\ t \in [0,T]\ :\quad [H_t\,,\, S_n(t)] \;=\; S_{n+1}(t)\,. \tag{III.66}$$

This in turn gives

$$\dot S_1 - \dot R_1 \;=\; [S_2, A^{-1}] - [\dot H, A^{-1}] \;=\; \big[S_2 - [H, S_1]\,,\, A^{-1}\big] \;=\; 0\,, \tag{III.67}$$

hence $S_1(t) = R_1(t) = [H_t, A^{-1}]$ and

$$\dot H_t \;=\; \big[H_t\,,\, [H_t, A^{-1}]\big]\,, \tag{III.68}$$

for all $t > 0$. In other words: $H_{(\cdot)}$ is a smooth solution of the original evolution equation (III.23).

(ii) Let $\widetilde H_{(\cdot)} \in C^\infty\big([0,T]; \mathcal{B}[\mathscr{D}, \mathscr{H}]\big)$ be another solution of (III.23), possibly different from $H_{(\cdot)}$ found in (i). We denote $R_n := R_n(H_t)$, as before, and $\widetilde R_n := R_n(\widetilde H_t)$. According to Lemma 3, both $\underline{R} = (R_n)_{n=0}^\infty$ and $\underline{\widetilde R} = (\widetilde R_n)_{n=0}^\infty$ solve (III.28). By the uniqueness of its solution, due to the fact that $\mathcal{F}$ is a contraction on B, we have $\underline{\widetilde R} = \underline{R}$ and in particular $\widetilde R_1 = R_1 = S_1$. This, in turn, implies that

$$\dot H - \dot{\widetilde H} \;=\; [H, R_1] \;-\; [\widetilde H, \widetilde R_1] \;=\; [H - \widetilde H\,,\, S_1] \tag{III.69}$$

and hence

$$\big\|(\dot H - \dot{\widetilde H})A^{-1}\big\|_X \;\leq\; \big\|(H - \widetilde H)A^{-1}\big\|_X\,\|S_1\|_Y \;+\; \|S_1\|_X\,\big\|(H - \widetilde H)A^{-1}\big\|_X\,. \tag{III.70}$$

It follows from $T < T_* = e^{\rho-\eta-1}/8$ and (III.54) that

$$\begin{aligned}\max_{0\leq t\leq T}\big\|(H_t - \widetilde H_t)A^{-1}\big\|_X \;&\leq\; 2\,T\,\|S_1\|_{X\cap Y}\,\max_{0\leq t\leq T}\big\|(H - \widetilde H)A^{-1}\big\|_X \\ &\leq\; \frac{1}{2}\,\max_{0\leq t\leq T}\big\|(H - \widetilde H)A^{-1}\big\|_X\,,\end{aligned} \tag{III.71}$$

which finally implies $\widetilde H_t = H_t$, for all $t \in [0,T]$. □

IV Convergence on Hilbert-Schmidt Operators

In Sect. I it was assumed that $H = H^*, A = A^* \in \mathrm{Mat}(N, \mathbb{C})$ are two self-adjoint complex $N \times N$-matrices such that $A \geq 0$.

We generalize this now and study Hilbert-Schmidt operators $H = H^*, A = A^* \in \mathcal{L}^2[\mathscr{H}]$, with $A \geq 0$, on a separable Hilbert space $\mathscr{H}$. We define the initial value $H_0 := H$ and study the flow equation

$$\forall t > 0: \qquad \dot{H}_t \; := \; i[H_t, G_t], \qquad G_t \; := \; i[A, H_t]. \tag{IV.72}$$

We use the fact proved in Section II that (IV.72) possesses a unique smooth global solution $H_{(\cdot)} \in C^\infty\big(\mathbb{R}_0^+; \mathcal{L}^2[\mathscr{H}]\big)$. This follows from an application of Theorem 1, taking into account that the norm $\|M\|_{\mathrm{hs}} = \sqrt{\mathrm{Tr}\{M^*M\}}$ on $\mathcal{L}^2[\mathscr{H}]$ is unitarily invariant.

Lemma 6 *Suppose $H_0 := H = H^*, A = A^* \in \mathcal{L}^2[\mathscr{H}]$ are two self-adjoint Hilbert-Schmidt operators on a separable Hilbert space $\mathscr{H}$ such that $A \geq 0$, and let $H_{(\cdot)} \in C^\infty\big(\mathbb{R}_0^+; \mathcal{L}^2[\mathscr{H}]\big)$ be the unique solution of* (IV.72). *Then*

$$\lim_{t \to \infty} i[A, H_t] \; = \; 0. \tag{IV.73}$$

Proof: We introduce

$$\forall t \geq 0: \quad g_t \; := \; -\dot{f}_t \; = \; \mathrm{Tr}\{G_t^2\} \; \geq \; 0 \tag{IV.74}$$

and observe that g is integrable, namely

$$\int_0^\infty g_t \, dt \; = \; f_0 - f_\infty \; \leq \; f_0. \tag{IV.75}$$

The integrability of g alone, however, does not imply that $g(t) \to 0$, as $t \to \infty$, since g could in principle have arbitrarily high bumps which are, yet, so narrow that they yield a small integral.

To conclude the convergence of g and then of H, we compute

$$\begin{aligned} \dot{g}(t) \; &= \; 2\,\mathrm{Tr}\{G_t \, \dot{G}_t\} \; = \; 2\,\mathrm{Tr}\{G_t \, [iA, \dot{H}_t]\} \\ &= \; 2\,\mathrm{Tr}\{\dot{H}_t \, [G_t, iA]\} \; = \; 2\,\mathrm{Tr}\{[iH_t, G_t] \, [-iA, G_t]\} \\ &\leq \; 2\,\mathrm{Tr}\{[i(H_t - A), G_t] \, [-iA, G_t]\} \; \leq \; 8\, \|A\| \, f_0^{1/2} \, g_t. \end{aligned} \tag{IV.76}$$

By Grønwall's Lemma, this differential inequality yields

$$\frac{g_t}{g_s} \; \leq \; \exp\big(8 \, \|A\| \, f_0^{1/2}(t-s)\big), \tag{IV.77}$$

for all $t \geq s \geq 0$. Applying this estimate to $s := n \in \mathbb{N}_0$ and $t \in [n, n+1]$, we obtain

$$\sum_{n=0}^{\infty} g_n \ \leq \ e^{8\|A\|\sqrt{f_0}} \int_0^{\infty} g_t \, dt \ \leq \ f_0 \, e^{8\|A\|\sqrt{f_0}}, \tag{IV.78}$$

from which we conclude the existence of a constant $C < \infty$, such that

$$g_n \ \leq \ \frac{C}{n+1}. \tag{IV.79}$$

Applying Estimate IV.77 again, but with $t := n \in \mathbb{N}$ and $s \in [n-1, n]$, we obtain

$$\forall\, s \geq 0 : \quad g_s \ \leq \ \frac{C\, e^{8\|A\|\sqrt{f_0}}}{s}, \tag{IV.80}$$

and therefore, we arrive at the assertion thanks to

$$\forall t \geq 0 : \qquad \mathrm{Tr}\{[iA, H_t]^2\} \ = \ g_t \ \leq \ \frac{C\, e^{8\|A\|\sqrt{f_0}}}{t}. \tag{IV.81}$$

□

In the next theorem we prove the convergence of H_t, as $t \to \infty$, under the additional assumption that $A \geq 0$ has full rank. To be more explicit, we assume that $A = \sum_{j=1}^{\infty} \alpha_j Q_j$, where $Q_j = Q_j^2 = Q_j^*$ are orthogonal projections of rank $n_j = \mathrm{Tr}\{Q_j\} \in \mathbb{N}$ and $\alpha_1 > \alpha_2 > \ldots > 0$, with $\mathrm{Tr}\{A^2\} = \sum_{j=1}^{\infty} \alpha_j^2 n_j < \infty$.

Theorem 7 *Suppose $H_0 \equiv H = H^*, A = A^* \in \mathcal{L}^2[\mathscr{H}]$ are two self-adjoint Hilbert-Schmidt operators on a separable Hilbert space $\mathscr{H}$ such that $A > 0$ has full rank. Let $H_{(\cdot)} \in C^{\infty}\big(\mathbb{R}_0^+; \mathcal{L}^2[\mathscr{H}]\big)$ be the unique solution of* (IV.72). *Then*

$$\lim_{t\to\infty} H_t \ =: \ H_{\infty} \tag{IV.82}$$

converges strongly on $\mathscr{H}$, we have $[H_{\infty}, A] = 0$, and there exists a unitary operator $W \in \mathcal{B}[\mathscr{H}]$ such that

$$H_{\infty} \ = \ W\, H_0\, W^*. \tag{IV.83}$$

Proof: As mentioned above, we assume without loss of generality that $A = \sum_{j=1}^{\infty} \alpha_j Q_j$, where $Q_j = Q_j^2 = Q_j^*$ are orthogonal projections of rank $n_j = \mathrm{Tr}\{Q_j\} \in \mathbb{N}$ and $\alpha_1 > \alpha_2 > \ldots > 0$, with $\mathrm{Tr}\{A^2\} = \sum_{j=1}^{\infty} \alpha_j^2 n_j < \infty$.

We set $\alpha_0 := \infty$ and

$$\kappa_j \ := \ \mathrm{dist}\big[\alpha_j\, ,\ \sigma(A) \setminus \{\alpha_j\}\big] \ = \ \min\big\{\alpha_{j-1} - \alpha_j\, ,\ \alpha_j - \alpha_{j+1}\big\} \ > \ 0, \tag{IV.84}$$

for all $j \in \mathbb{N}$. We first observe, for $j \in \mathbb{N}$, that due to (IV.74), we have

$$\begin{aligned} \mathrm{Tr}\{Q_j H_t Q_j^\perp H_t Q_j\} &= \sum_{k=1}^{\infty} \big(1 - \delta_{j,k}\big) \; \mathrm{Tr}\{Q_j H_t Q_k H_t Q_j\} \\ &\le \frac{1}{\kappa_j^2} \sum_{k,\ell=1}^{\infty} \mathrm{Tr}\{Q_\ell H_t Q_k H_t Q_\ell\} \, (\alpha_\ell - \alpha_k)^2 \\ &= \frac{1}{\kappa_j^2} \; \mathrm{Tr}\left\{ [iA, H_t]^2 \right\} \; = \; \frac{g_t}{\kappa_j^2} \, , \end{aligned} \tag{IV.85}$$

which shows that the off-diagonal matrix blocks $Q_j H_t Q_j^\perp$ and $Q_j^\perp H_t Q_j$ of H_t tend to 0, as $t \to \infty$. Next, we prove the convergence of the diagonal matrix blocks $Q_j H_t Q_j$. To this end, we observe that, for $1 \le j < \infty$,

$$\begin{aligned} \big\| Q_j \dot{H}_t Q_j \big\|_{\mathcal{L}^1[\mathscr{H}]} &= \big\| Q_j H_t G_t Q_j \; - \; Q_j G_t H_t Q_j \big\|_{\mathcal{L}^1[\mathscr{H}]} \\ &= \big\| Q_j H_t Q_j^\perp G_t Q_j \; - \; Q_j G_t Q_j^\perp H_t Q_j \big\|_{\mathcal{L}^1[\mathscr{H}]} \\ &\le 2 \big\| Q_j H_t Q_j^\perp G_t Q_j \big\|_{\mathcal{L}^1[\mathscr{H}]} \\ &= 2 \Big\| \sum_{k=1}^{\infty} Q_j H_t Q_k H_t Q_j \, (\alpha_j - \alpha_k) \Big\|_{\mathcal{L}^1[\mathscr{H}]} \\ &\le 2 \sum_{k=1}^{\infty} \mathrm{Tr}\{Q_j H_t Q_k H_t Q_j\} \, |\alpha_j - \alpha_k| \\ &\le \frac{2}{\kappa_j} \sum_{k,\ell=1}^{\infty} \mathrm{Tr}\{Q_\ell H_t Q_k H_t Q_\ell\} \, (\alpha_\ell - \alpha_k)^2 \\ &= \frac{2}{\kappa_j} \; \mathrm{Tr}\left\{ [iA, H_t]^2 \right\} \; = \; \frac{2 \, g_t}{\kappa_j}, \end{aligned} \tag{IV.86}$$

using that $Q_j G_t = Q_j G_t Q_j^\perp$ and $Q_j H_t Q_k H_t Q_j \ge 0$, where the latter implies that $\|Q_j H_t Q_k H_t Q_j\|_{\mathcal{L}^1[\mathscr{H}]} = \mathrm{Tr}\{Q_j H_t Q_k H_t Q_j\}$. Since g_t is integrable, so is $\|Q_j \dot{H}_t Q_j\|_{\mathcal{L}^1[\mathscr{H}]}$, which establishes the desired convergence of the diagonal blocks $Q_j H_t Q_j$ in $\mathcal{L}^1[\mathscr{H}]$. It follows that H_t converges strongly, i.e., (IV.82).

We proceed to proving (IV.83). Given $H_\infty := \lim_{t\to\infty} H_t$, we invoke the spectral theorem to obtain its spectral decomposition

$$H_\infty \; =: \; \sum_{\ell=1}^{L} \lambda_\ell \, P_\ell, \tag{IV.87}$$

where $\lambda_1 < \lambda_2 < \cdots < \lambda_L$ are its eigenvalues and $P_1, P_2, \ldots, P_L$ are its spectral projections (possibly $L = \infty$), i.e., orthogonal projections P_ℓ of rank

$n_\ell \geq 1$ (possibly, $n_\ell = \infty$, too). Choosing $r > 0$ sufficiently small, these projections may be written as the Cauchy integral

$$P_\ell \;=\; \frac{1}{2\pi i}\int_{|z|=r} \frac{dz}{z - H_\infty}. \tag{IV.88}$$

Next, define the simplex $\mathcal{S} := \{(T,t) \in \mathbb{R}^2 | 0 \leq t \leq T\}$ and let $(U_{T,t})_{T\geq t\geq 0} \in C^\infty\big(\mathcal{S}; \mathcal{B}[\mathscr{H}]\big)$ be the unique unitary solution of

$$\forall T > t \geq 0: \qquad \partial_T U_{T,t} \;=\; -iG_T\, U_{T,t}, \qquad U_{t,t} \;=\; \mathbf{1}. \tag{IV.89}$$

Note the cocyclicity $U_{T,t}U_{t,s} = U_{T,s}$, for $T \geq t \geq s \geq 0$. This property implies that

$$\forall T > t \geq 0: \qquad H_T \;=\; U_{T,0}\, H_0\, U_{T,0}^* \;=\; U_{T,t}\, H_t\, U_{T,t}^* \;\to\; H_\infty, \tag{IV.90}$$

as $T \to \infty$, from which we obtain

$$\forall t \geq 0: \qquad \lim_{T\to\infty} \big\{U_{T,t}^*\, H_\infty\, U_{T,t}\big\} \;=\; H_t. \tag{IV.91}$$

We then define $P_\ell(t)$ by

$$P_\ell(t) \;:=\; \mathbf{1}\big(|H_t - \lambda_\ell| < r\big) \;=\; \frac{1}{2\pi i}\int_{|z|=r} \frac{dz}{z - H_t}, \tag{IV.92}$$

for $t > 0$ and sufficiently small $r > 0$. Due to (IV.91), we have

$$P_\ell(t) \;=\; \lim_{T\to\infty} \big\{U_{T,t}^*\, P_\ell\, U_{T,t}\big\}. \tag{IV.93}$$

In particular, the rank of $P_\ell(t)$ is bounded by

$$\mathrm{rk}\{P_\ell(t)\} \;\leq\; n_\ell. \tag{IV.94}$$

Moreover,

$$\sum_{\ell=1}^{L} P_\ell(t) \;=\; \lim_{T\to\infty} \Big\{U_{T,t}^* \big(\sum_{\ell=1}^{L} P_\ell\big) U_{T,t}\Big\} \;=\; \mathbf{1}. \tag{IV.95}$$

Eqs. (IV.94) and (IV.95) yield $\mathrm{rk}\{P_\ell(t)\} = n_\ell$ and hence

$$H_t \;=\; \sum_{\ell=1}^{L} \lambda_\ell\, P_\ell(t) \tag{IV.96}$$

is the spectral decomposition of H_t. For each ℓ, we can now pick orthonormal bases $\{\varphi_n | 1 \leq n \leq n_\ell\}$ and $\{\psi_n | 1 \leq n \leq n_\ell\}$ of $\mathrm{Ran}\, P_\ell(0)$ and $\mathrm{Ran}\, P_\ell$, respectively. We then define a unitary $W \in \mathcal{B}[\mathscr{H}]$ by

$$\forall\, n \geq 1: \qquad W\varphi_n \;:=\; \psi_n, \tag{IV.97}$$

and we clearly have

$$W\, H_0\, W^* \;=\; H_\infty. \tag{IV.98}$$

□

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