On second order perturbation theory for embedded eigenvalues

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Mathematical models of QFT
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Outline of the talk

1. Regular Mourre theory with a self-adjoint conjugate operator

2. The Nelson model

3. Singular Mourre theory with a non self-adjoint conjugate operator

Part I

Regular Mourre theory with a self-adjoint conjugate operator

Regularity w.r.t. a self-adjoint operator

- $\mathcal{H}$ complex Hilbert space
- H, A self-adjoint operators on $\mathcal{H}$

Definition

Let $n \in \mathbb{N}$. We say that $H \in C^n(A)$ if and only if $\forall z \in \mathbb{C} \setminus \sigma(H)$, $\forall \phi \in \mathcal{H}$,

$$s \mapsto e^{isA}(H-z)^{-1}e^{-isA}\phi \in C^n(\mathbb{R})$$

Remarks

- $H \in C^1(A)$ if and only if $\forall z \in \mathbb{C} \setminus \sigma(H)$, $(H-z)^{-1}D(A) \subseteq D(A)$, and $\forall \phi \in D(H) \cap D(A)$,

$$|\langle A\phi, H\phi\rangle - \langle H\phi, A\phi\rangle| \leq C(\|H\phi\|^2 + \|\phi\|^2)$$

- If $H \in C^1(A)$, then $D(H) \cap D(A)$ is a core for H, and the quadratic form $[H, A]$ defined on $(D(H) \cap D(A)) \times (D(H) \cap D(A))$ extend by continuity to a bounded quadratic form on $D(H) \times D(H)$ denoted $[H, A]^0$

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Mourre estimate

Definition

Let I be a bounded open interval, $\mathrm{I} \subset \sigma(H)$. We say that H satisfies a Mourre estimate on I with A as conjugate operator if $\exists\, c_0 > 0$ and K_0 compact such that

$$\mathbb{1}_{\mathrm{I}}(H)[H, iA]^0 \mathbb{1}_{\mathrm{I}}(H) \geq c_0 \mathbb{1}_{\mathrm{I}}(H) - K_0,$$

in the sense of quadratic forms on $\mathcal{H} \times \mathcal{H}$

Remarks

- An equivalent formulation is

$$[H, iA]^0 \geq c_0' - c_1' \mathbb{1}_{\mathbb{R}\setminus\mathrm{I}}(H)\langle H\rangle - K_0',$$

in the sense of quadratic forms on $D(H) \times D(H)$, with $c_0' > 0$, $c_1' \in \mathbb{R}$, and K_0' compact

- If $K_0 = 0$, we say that H satisfies a strict Mourre estimate on I

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The Virial Theorem

Theorem ([Mo '81], [ABG '96], [GG '99])

Let ϕ be an eigenstate of H. If $H \in C^1(A)$, then

$$\langle \phi, [H, iA]^0 \phi \rangle = 0$$

Corollary

Assume that $H \in C^1(A)$ and that H satisfies a Mourre estimate on I. Then the number of eigenvalues of H in I is finite, and each such eigenvalue has a finite multiplicity

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Limiting Absorption Principle

Theorem ([Mo '81], [ABG '96], [Ge '08])

Assume that $H \in C^2(A)$ and that H satisfies a strict Mourre estimate on I. Then for all closed interval $J \subset \mathrm{I}$ and $s > 1/2$,

$$\sup_{z \in J^\pm} \| \langle A \rangle^{-s} (H - z)^{-1} \langle A \rangle^{-s} \| < \infty,$$

where $J^\pm = \{ z \in \mathbb{C}, \mathrm{Re}\, z \in J, \pm \mathrm{Im}\, z > 0 \}$ and $\langle A \rangle = (1 + A^2)^{1/2}$. In particular the spectrum of H in I is purely absolutely continuous.
Moreover for $1/2 < s \leq 1$, the maps

$$J^\pm \ni z \mapsto \| \langle A \rangle^{-s} (H - z)^{-1} \langle A \rangle^{-s} \| \in B(\mathcal{H})$$

are Hölder continuous of order $s - 1/2$. In particular, for $\lambda \in J$, the limits

$$\langle A \rangle^{-s} (H - \lambda \pm i0)^{-1} \langle A \rangle^{-s} := \lim_{\epsilon \downarrow 0} \langle A \rangle^{-s} (H - \lambda \pm i\epsilon)^{-1} \langle A \rangle^{-s}$$

exist in the norm topology of $B(\mathcal{H})$, and the corresponding functions of λ are Hölder continuous of order $s - 1/2$

Fermi Golden Rule criterion

Theorem ([AHS '89], [HuSi '00])

Suppose

1) (Regularity of H) $H \in C^2(A)$ and the quadratic forms $[H, iA]$ and $[[H, iA], iA]$ extend by continuity to H-bounded operators
2) (Mourre estimate) H satisfies a Mourre estimate on I

Let $\lambda \in \mathrm{I}$ be an eigenvalue of H. Let $P = \mathbb{1}_{\{\lambda\}}(H)$ be the associated eigenprojection and $\bar{P} = I - P$. Let $J \subset \mathrm{I}$ be a closed interval such that $\sigma_{\mathrm{pp}}(H) \cap J = \{\lambda\}$. Let W be a symmetric and H-bounded operator. Suppose

3) (Regularity of eigenstates) $\mathrm{Ran}(P) \subseteq D(A^2)$
4) (Regularity of the perturbation) $[W, iA]$ and $[[W, iA], iA]$ extend by continuity to H-bounded operators

If the Fermi Golden Rule criterion is satisfied, i.e.

$$PW\mathrm{Im}((H - \lambda - i0)^{-1}\bar{P})WP \geq cP$$

with $c > 0$, then $\exists\, \sigma_0 > 0$ such that $\forall\, 0 < |\sigma| \leq \sigma_0$,

$$\sigma_{\mathrm{pp}}(H + \sigma W) \cap J = \emptyset$$

Regularity of bound states

Theorem ([Ca '05], [CGH '06])

Let $n \in \mathbb{N}$. Assume that $H \in C^{n+2}(A)$ and that $\mathrm{ad}_A^k(H)$ are H-bounded for all $1 \leq k \leq n+2$. Assume that H satisfies a Mourre estimate on I. Let $\lambda \in \mathrm{I}$ be an eigenvalue of H and let $P = \mathbb{1}_{\{\lambda\}}(H)$ be the associated eigenprojection. Then we have that

$$\mathrm{Ran}(P) \subseteq D(A^n)$$

Remark

In fact $H \in C^{n+1}(A)$ is sufficient for the conclusion of the previous theorem to hold and this is optimal ([FMS' 10], [MW' 10]).

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Part II

The Nelson model

Definition of the model

- Hilbert space: $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^3) \otimes \mathcal{F} \simeq \mathrm{L}^2(\mathbb{R}^3; \mathcal{F})$ where $\mathcal{F}$ is the symmetric Fock space over $\mathrm{L}^2(\mathbb{R}^3)$ defined by $\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n=1}^{+\infty} \mathrm{L}^2(\mathbb{R}^3)^{\otimes_s^n}$

- Hamiltonian: $H_g = H_{\mathrm{el}} \otimes \mathbb{1} + \mathbb{1} \otimes H_f + g\phi(h(x))$ where

  - $$H_{\mathrm{el}} = -\Delta + V(x) + U(x)$$
    with $V \ll \Delta$ and $U(x) \geq c_0|x|^\alpha - c_1$, $c_0 > 0$, $\alpha > 4$

  - $$H_f = \mathrm{d}\Gamma(|k|)$$

  - $$\phi(h(x)) = a^*(h(x)) + a(h(x))$$
    where $\forall x \in \mathbb{R}^3$, $h(x) \in \mathrm{L}^2(\mathbb{R}^3, \mathrm{d}k)$ is given by
    $$h(x,k) = \frac{\chi(k)}{|k|^{\frac{1}{2}-\epsilon}} e^{-ik\cdot x}, \quad \chi \in \mathrm{C}_0^\infty(\mathbb{R}^3), \quad \epsilon > 0$$

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Fermi Golden Rule

- Let H_0 be the 'unperturbed' operator. Under different assumptions, it is established that, for sufficiently small values of g, Fermi Golden Rule holds for excited unperturbed eigenvalues ([BFS '99], [BFSS '99], [DJ '01], [Go '09]). In particular the spectrum of H_g is purely absolutely continuous in a neighborhood of the excited unperturbed eigenvalues

- Problem: show that 'generically' H_g does not have eigenvalue above the ground state energy for an arbitrary value of g. More precisely, assuming that λ is an eigenvalue of H_g for a given $g \in \mathbb{R}$, we want to show that λ is unstable under small perturbations according to Fermi Golden Rule

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Choice of the conjugate operator

- Generator of dilatations in Fock space

$$A_1 = \mathbb{1} \otimes \mathrm{d}\Gamma(a_1) = \mathbb{1} \otimes \mathrm{d}\Gamma(\frac{i}{2}(\nabla_k \cdot k + k \cdot \nabla_k))$$

Formal commutator with H_g:

$$[H_g, iA_1] = \mathrm{d}\Gamma(|k|) - g\phi(ia_1 h(x))$$

Difficulty when g is not supposed to be small

- Generator of radial translation in Fock space

$$A_2 = \mathbb{1} \otimes \mathrm{d}\Gamma(a_2) = \mathbb{1} \otimes \mathrm{d}\Gamma(\frac{i}{2}(\nabla_k \cdot \frac{k}{|k|} + \frac{k}{|k|} \cdot \nabla_k))$$

Formal commutator with H_g:

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Difficulties

- A_2 is not self-adjoint, only maximal symmetric. Mourre theory with a non self-adjoint conjugate operator initiated in [HüSp '95] (the conjugate operator is supposed to be the generator of a C_0-semigroup)
- $[H_g, iA_2]$ is not controlled by H_g (the quadratic form is not bounded on $D(H_g) \times D(H_g)$). This situation is referred to as 'singlular' Mourre theory ([Sk '98], [MS '03], [GGM '04])
- Each time we commute with iA_2, the singularity in the field operator is increased by a power of $|k|$. As far as the infrared singularity is concerned, it is crucial to minimize the number of commutators of H_g with A_2 we need to estimate

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Part III

Singular Mourre theory with a non self-adjoint conjugate operator

Framework

- $\mathcal{H}$ complex Hilbert space
- H, M self-adjoint operators, $M \geq 0$, $\mathcal{G} = D(M^{\frac{1}{2}}) \cap D(|H|^{\frac{1}{2}})$
- R symmetric operator, $D(R) \supseteq D(H)$
- A closed operator, densely defined, maximal symmetric. Assuming that A has deficiency indices $(N, 0)$, this implies that A generates a C_0-semigroup of isometries $\{W_t\}_{t \geq 0}$

Definition

The map $[0, \infty) \ni t \mapsto W_t \in B(\mathcal{H})$ is called a C_0-semigroup if $W_0 = I$, $W_t W_s = W_{t+s}$ and $w - \lim_{t \to 0} W_t = I$. The generator of a C_0-semigroup is defined by

$$D(A) = \left\{ u \in \mathcal{H}, Au := \lim_{t \to 0} \frac{1}{it}(W_t u - u) exists \right\}$$

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Regularity with respect to C_0-semigroups

Definition

Let $\{W_{1,t}\}$ and $\{W_{2,t}\}$ be two C_0-semigroups in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with generators A_1 and A_2 respectively. A bounded operator $B \in B(\mathcal{H}_1;\mathcal{H}_2)$ is said to be in $C^1(A_1, A_2)$ if

$$\|W_{2,t}B - BW_{1,t}\|_{B(\mathcal{H}_1;\mathcal{H}_2)} \leq Ct, \quad 0 \leq t \leq 1$$

Remarks

- $B \in C^1(A_1; A_2)$ iff the quadratic form defined on $D(A_2^*) \times D(A_1)$

$$i\langle B^*\phi, A_1\psi\rangle_{\mathcal{H}_1} - i\langle A_2^*\phi, B\psi\rangle_{\mathcal{H}_2}$$

  extends by continuity to a bounded quadratic form on $\mathcal{H}_2 \times \mathcal{H}_1$
- The bounded operator in $B(\mathcal{H}_1;\mathcal{H}_2)$ associated to the previous quadratic form is denoted by $[B, iA]^0$, and we have that

$$[B, iA]^0 = s - \lim_{t\to 0} \frac{1}{t}(BW_{1,t} - W_{2,t}B)$$

- If $B \in C^1(A_1; A_2)$ and $[B, iA]^0 \in C^1(A_1; A_2)$ we say that $B \in C^2(A_1; A_2)$

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Assumptions (I)

(Regularity of H with respect to A)

- $W_t\mathcal{G} \subseteq \mathcal{G}$, $W_t^*\mathcal{G} \subseteq \mathcal{G}$, and $\forall \phi \in \mathcal{G}$,

$$\sup_{0<t<1} \|W_t\phi\| < \infty, \quad \sup_{0<t<1} \|W_t^*\phi\| < \infty$$

  This implies that
  $*$ $W_t|_{\mathcal{G}}$ is a C_0-semigroup whose generator is denoted by $A_{\mathcal{G}}$
  $*$ W_t extends to a C_0-semigroup in $\mathcal{G}^*$ whose generator is denoted by $A_{\mathcal{G}^*}$
- $H \in C^2(A_{\mathcal{G}}; A_{\mathcal{G}^*})$ and for all $\phi \in D(H) \cap D(M)$,

$$[H, iA]^0\phi = (M + R)\phi$$

(Regularity of H with respect to M)

$H \in C^1(M)$ and $[H, iM]^0$ is H-bounded

Assumptions (I)

(Regularity of H with respect to A)

- $W_t\mathcal{G} \subseteq \mathcal{G}$, $W_t^*\mathcal{G} \subseteq \mathcal{G}$, and $\forall \phi \in \mathcal{G}$,

$$\sup_{0<t<1} \|W_t\phi\| < \infty, \qquad \sup_{0<t<1} \|W_t^*\phi\| < \infty$$

  This implies that
  $*$ $W_t|_{\mathcal{G}}$ is a C_0-semigroup whose generator is denoted by $A_{\mathcal{G}}$
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The Virial Theorem

Remark

Under the previous assumptions,

$$\langle \phi_1, (M+R)\phi_2 \rangle = i\langle H\phi_1, A\phi_2 \rangle - i\langle A^*\phi_1, H\phi_2 \rangle$$

for all $\phi_1 \in D(H) \cap D(M) \cap D(A^*)$ and $\phi_2 \in D(H) \cap D(M) \cap D(A)$

Theorem ([GGM '04])

Assume that the previous hypotheses hold. If ψ is an eigenstate of H such that $\psi \in D(M^{\frac{1}{2}})$, then

$$\langle \psi, (M+R)\psi \rangle := \|M^{\frac{1}{2}}\psi\|^2 + \langle \psi, R\psi \rangle = 0$$

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Assumptions (II)

(Mourre estimate)

$\exists$ an interval $\mathrm{I} \subseteq \mathbb{R}$ such that $\forall \eta \in \mathrm{I}$, $\exists\, c_0 > 0$, $C_1 \in \mathbb{R}$, K_0 compact, and a function $f_\eta \in C_0^\infty(\mathbb{R};[0,1])$ such that $f_\eta = 1$ in a neighborhood of η and

$$M + R \geq c_0 - C_1 f_\eta^\perp(H)^2 \langle H \rangle - K_0,$$

in the sense of quadratic forms on $D(H) \cap D(M)$, where $f_\eta^\perp = 1 - f_\eta$

(Regularity of bound states and the perturbation) $(*)$

For all compact interval $J \subseteq \mathrm{I}$, $\exists \gamma > 0$ and a set B_γ such that

$$B_\gamma \subseteq \{ V \text{ symmetric and } H\text{-bounded}, V \in C^1(A_{\mathcal{G}}; A_{\mathcal{G}^*}) \\ \|V\|_1 := \|V(H-i)^{-1}\| + \|[V, iA]^0 (H-i)^{-1}\| \leq \gamma \},$$

$\{0\} \subset B_\gamma$, B_γ is star-shaped and symmetric w.r.t. 0, and the following holds: $\exists\, C > 0$, $\forall V \in B_\gamma$, $\forall \lambda \in J$, $\forall \psi \in D(H)$, $(H + V - \lambda)\psi = 0$, we have that

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Upper semicontinuity of point spectrum

Theorem ([FMS' 10])

Assume that the previous hypotheses hold. Let $J \subseteq \mathrm{I}$ be a compact interval such that $\sigma_{\mathrm{pp}}(H) \cap J = \{\lambda\}$. There exists $0 < \gamma' \leq \gamma$ such that if $V \in B_\gamma$ and $\|V\|_1 \leq \gamma'$, then the total multiplicity of the eigenvalues of $H + V$ in J is at most $\dim \mathrm{Ker}(H - \lambda)$

Remark

In the case where $\sigma_{\mathrm{pp}}(H) \cap J = \emptyset$, Hypothesis $(*)$ on the regularity of bound states and the perturbation is not necessary to conclude that $\sigma_{\mathrm{pp}}(H + V) \cap J = \emptyset$. It is sufficient to assume that

- $V \in C^2(A_{\mathcal{G}}; A_{\mathcal{G}^*})$ and V, $[V, iA]^0$ are H-bounded

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- $V \in C^1(A_{\mathcal{G}}; A_{\mathcal{G}^*})$, V and $[V, iA]^0$ are H-bounded, and the possibly existing eigenstates of $H + V$ belong to $D(M^{\frac{1}{2}})$

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Fermi Golden Rule criterion

(Further technical hypothesis)

$D(M^{\frac{1}{2}}) \cap D(H) \cap D(A^*)$ is a core for A^*

Theorem ([FMS '10])

Assume that the previous hypotheses hold. Let $J \subseteq \mathrm{I}$ be a compact interval such that $\sigma_{\mathrm{pp}}(H) \cap J = \{\lambda\}$. Let $P = \mathbb{1}_{\{\lambda\}}(H)$ and $\bar{P} = I - P$. Let $V \in B_\gamma$ be such that

$$PV\mathrm{Im}\big((H - \lambda - i0)^{-1}\bar{P}\big)VP \geq cP, \quad c > 0.$$

There exists $\sigma_0 > 0$ such that for all $0 < |\sigma| \leq \sigma_0, \quad \sigma_{\mathrm{pp}}(H + \sigma V) \cap J = \emptyset$

Remark

Hypothesis $(*)$ on the regularity of bound states and the perturbation can be replaced by the following two assumptions:

- $\mathrm{Ran}(P) \subseteq D(A^2)$
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Second order expansion of eigenvalues (simple case)

Theorem ([FMS '10])

Assume that the previous hypotheses hold. Let $J \subseteq \mathrm{I}$ be a compact interval such that $\sigma_{\mathrm{pp}}(H) \cap J = \{\lambda\}$. Let $P = \mathbb{1}_{\{\lambda\}}(H)$ and $\bar{P} = I - P$. Let $V \in B_\gamma$. Suppose that

$$P = |\psi\rangle\langle\psi|.$$

For all $\epsilon > 0$, there exists $\sigma_0 > 0$ such that if $|\sigma| \le \sigma_0$ and $\lambda_\sigma \in J$ is an eigenvalue of $H + \sigma V$, then

$$\left|\lambda_\sigma - \lambda - \sigma\langle\psi, V\psi\rangle + \sigma^2\langle V\psi, (H - \lambda - i0)^{-1}\bar{P}V\psi\rangle\right| \le \epsilon\sigma^2,$$

and there exists a normalized eigenstate ψ_σ, $H_\sigma\psi_\sigma = \lambda_\sigma\psi_\sigma$, such that

$$\left\|\psi_\sigma - \psi + \sigma(H - \lambda - i0)^{-1}\bar{P}V\psi\right\|_{D(A)^*} \le \epsilon|\sigma|$$

Second order expansion of eigenvalues (general case)

If Hypothesis $(*)$ on the regularity of bound states and the perturbation is replaced by the following two assumptions:

- $\mathrm{Ran}(P) \subseteq D(A^2)$
- $V \in C^2(A_{\mathcal{G}}; A_{\mathcal{G}^*})$ and V, $[V, iA]^0$ are H-bounded

then the following theorem holds:

Theorem ([FMS '10])

Let $J \subseteq \mathrm{I}$ be a compact interval such that $\sigma_{\mathrm{pp}}(H) \cap J = \{\lambda\}$. Let $P = \mathbb{1}_{\{\lambda\}}(H)$ and $\bar{P} = I - P$. There exist $C \geq 0$ and $\sigma_0 > 0$ such that if $|\sigma| \leq \sigma_0$ and $\lambda_\sigma \in J$ is an eigenvalue of $H_\sigma = H + \sigma V$, then there exists $\psi \in \mathrm{Ran}(P)$, $\|\psi\| = 1$, such that

$$\left|\lambda_\sigma - \lambda - \sigma\langle\psi, V\psi\rangle + \sigma^2\langle V\psi, (H - \lambda - i0)^{-1}\bar{P}V\psi\rangle\right| \leq C|\sigma|^{\frac{5}{2}}$$