

Thermal relaxation in a quantum cavity

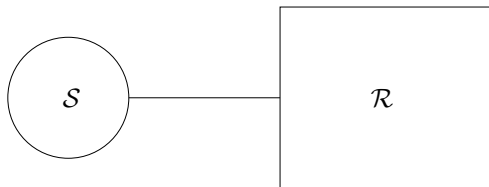
(Collaboration with C.A. Pillet)

L. Bruneau

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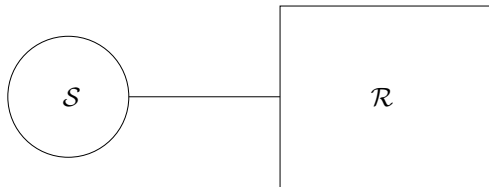
Mathematical models of Quantum Field Theory, Ecole Polytechnique,
07-08 December 2010

A “small” (or confined) system \mathcal{S} interacts with an environment \mathcal{R} .



Goal: understand the asymptotic ($t \rightarrow +\infty$) behaviour of the system \mathcal{S} (asymptotic state, thermodynamical properties).

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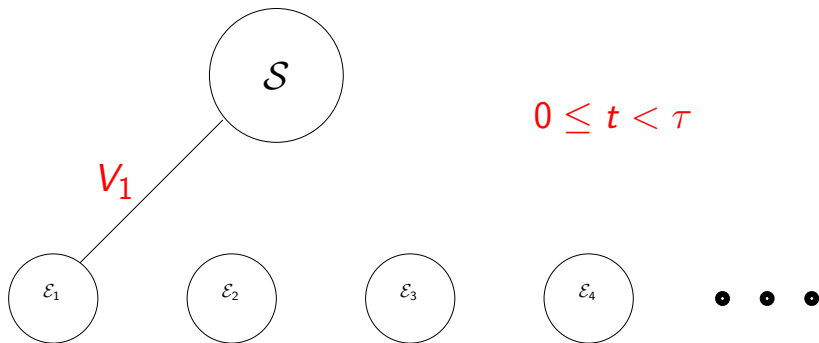
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2 approaches: Hamiltonian / Markovian

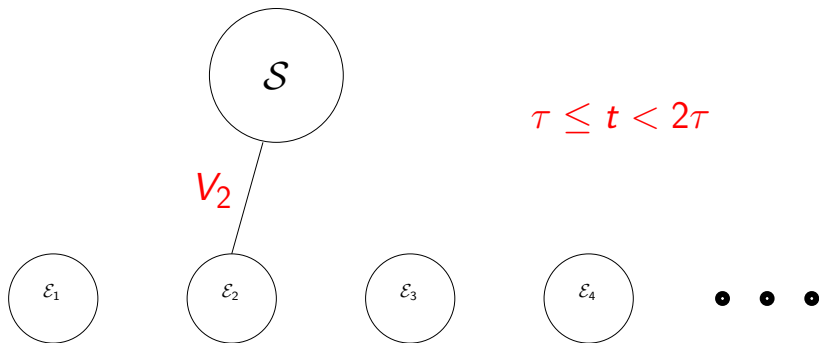
- **Hamiltonian:** full description, spectral analysis, scattering theory.
Restrictions: perturbative results, \mathcal{S} finite dimensional.
- **Markovian:** effective description of \mathcal{S} , obtained by weak-coupling type limits or if \mathcal{S} undergoes stochastic forces (Langevin equation).

Repeated Interaction Quantum Systems (RIQS)

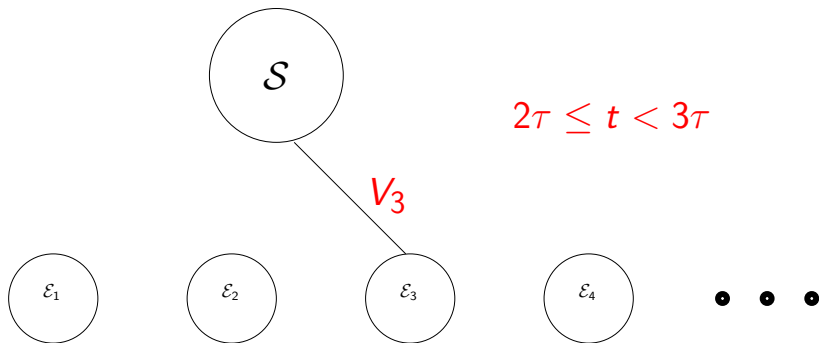
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For $t \in [(n-1)\tau, n\tau[$:

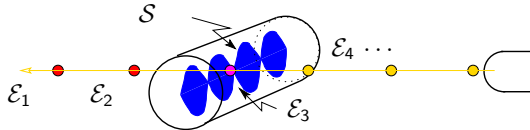
- \mathcal{S} interacts with \mathcal{E}_n ,
- \mathcal{E}_k evolves freely for $k \neq n$,

i.e. the full system is governed by

$$\widetilde{H}_n = H_S + H_{\mathcal{E},n} + V_n + \sum_{k \neq n} H_{\mathcal{E},k} = H_n + \sum_{k \neq n} H_{\mathcal{E},k}.$$

Some motivations

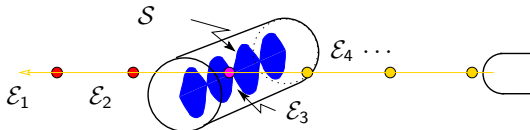
- Physics: “One-atom masers” (Walther et al '85, Haroche et al '92)



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 - \mathcal{E}_k = k -th atom interacting with the field.
 - \mathcal{C} : beam of atoms sent into the cavity.
- 2 Mathematics: Because of their particular structure (they are both Hamiltonian and Markovian), develop our understanding of open quantum systems, e.g. small system of infinite dimension, large coupling constant.

Mathematical model of the one-atom maser

- 1 The field in the cavity: (an harmonic oscillator)

$$\mathcal{H}_S = \Gamma_s(\mathbb{C}), \quad H_S = \omega a^* a = \omega N.$$

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This is the Jaynes-Cummings hamiltonian (dipole interaction in the rotating-wave approximation).

The repeated interaction dynamics.

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After **1** interaction, the state of the total system is

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The reduced dynamics map

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Main difficulty: Perturbation theory doesn't work.

When $\lambda = 0$, $\mathcal{L}_\beta(\rho) = e^{-i\tau H_S} \rho e^{i\tau H_S}$. Hence

$\text{sp}(\mathcal{L}_\beta) = \{e^{i\omega\tau(n-m)}, n, m \in \mathbb{Z}\}$: pure point spectrum, but all the eigenvalues, and in particular 1, are infinitely degenerate!

Jaynes-Cummings Hamiltonian and Rabi oscillations

If there are n photons in the cavity, the probability for the atom to make a transition $|-\rangle \rightarrow |+\rangle$ is a periodic function of time

$$P(t) = |\langle n-1, + | e^{-itH} | n, - \rangle| = \left(1 - \frac{\Delta^2}{\nu_n^2}\right) \sin^2\left(\frac{\nu_n t}{2}\right),$$

with frequency

$$\nu_n := \sqrt{\lambda^2 n + (\omega - \omega_0)^2} = \sqrt{\lambda^2 n + \Delta^2}.$$

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Conclusion: If the field is in state $|n\rangle$ before an interaction and τ is a multiple of the Rabi period $T_n := \frac{2\pi}{\nu_n}$, after this interaction it can not be in state $|n-1\rangle$: there is a decoupling between the “energy levels” $n-1$ and n .

Rabi resonances

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3 possible situations (depending on the arithmetic properties of ξ and η):
 $R(\xi, \eta)$ is empty, a singlet or infinite.

Generically: $R(\xi, \eta)$ is empty = no resonance. We now restrict (for the talk) to this non-resonant situation.

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- 3 exponentially mixing if there exists $\alpha > 0$ s.t. for any $\mu \ll \rho$, and any $A \in \mathcal{B}(\mathcal{H})$

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To understand the ergodic properties of \mathcal{L}_β , the main issue is to understand its peripheral spectrum, i.e. $\text{sp}(\mathcal{L}_\beta) \cap S^1$.

In particular, the invariant states are the possible ergodic states.

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• Use gauge symmetry: $[H, a^* a + b^* b] = [H_\mathcal{E}, \rho_\beta] = 0$

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- 3 A Perron-Frobenius type lemma (Shrader '2000) for completely positive maps on trace ideals \mathcal{J}_ρ :

$$\mathcal{L}_\beta(X) = e^{i\theta} X \Rightarrow \mathcal{L}_\beta(|X|) = |X| \text{ where } |X| = \sqrt{X^*X}.$$

Ergodicity and mixing (II)

Proposition

If $R(\xi, \eta) = \emptyset$, 1 is the only eigenvalue of \mathcal{L}_β on S^1 and it is simple. The unique invariant state is

$$\rho_S^{\beta^*} = \frac{e^{-\beta^* H_S}}{\text{Tr}(e^{-\beta^* H_S})}.$$

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Remarks:

- 1) There is a weak form of decoherence.
- 2) Numerically it seems that $\rho_S^{\beta^*}$ is not only ergodic but also mixing.
- 3) If $R(\xi, \eta) \neq \emptyset$ the multiplicity of 1 increases (one invariant state per “sector”).

Quasi-resonances

Recall: for diagonal states

$$\mathcal{L}_\beta = \mathbb{1} - \nabla^* D(N) e^{-\beta \omega_0 N} \nabla e^{\beta \omega_0 N}$$

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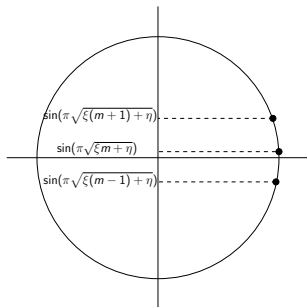
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We call $m \in \mathbb{N}^*$ a **quasi-resonance** if $D(m) < D(m \pm 1)$.



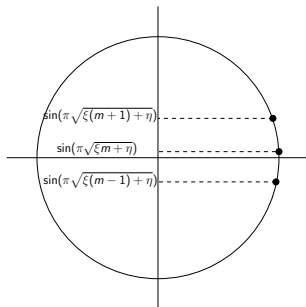
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If $(m_k)_k$ denotes the sequence of quasi-resonances, we have $D(m_k) = O(k^{-2})$.

Let $\mathcal{L}_\beta^0 = \mathbb{1} - \nabla^* D_0(N) e^{-\beta\omega_0 N} \nabla e^{\beta\omega_0 N}$ where

$$D_0(n) = \begin{cases} 0 & \text{if } n \in \{m_1, \dots\}, \\ D(n) & \text{otherwise.} \end{cases}$$

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Then $\mathcal{L}_\beta = \mathcal{L}_\beta^0 + \mathcal{T}$ where \mathcal{T} is of trace class and 1 is an infinitely degenerate eigenvalue of \mathcal{L}_β^0 .

\Rightarrow 1 **always** belongs to the essential spectrum of \mathcal{L}_β .

Metastable states

Let $\mathcal{L}_\beta^0 = \mathbb{1} - \nabla^* D_0(N) e^{-\beta\omega_0 N} \nabla e^{\beta\omega_0 N}$ where

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\Rightarrow 1 **always** belongs to the essential spectrum of \mathcal{L}_β .

The eigenstates of \mathcal{L}_β^0 are metastable states.

\Rightarrow There are infinitely many metastable states with arbitrarily large lifetimes. Hence we can **not** expect **exponential mixing**.

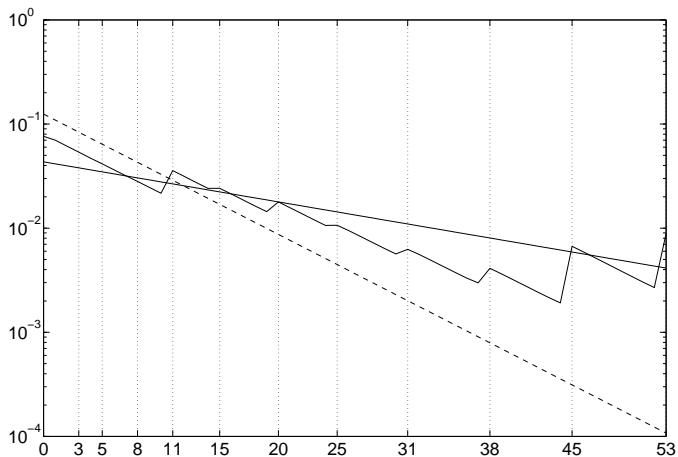


Figure: Cooling the cavity: 5000 interactions.

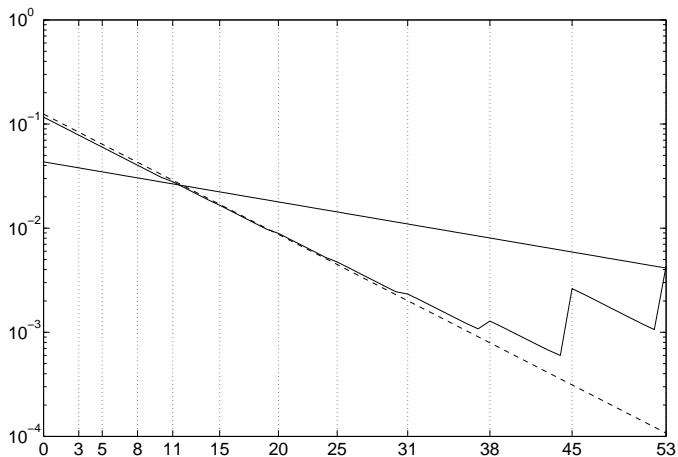


Figure: Cooling the cavity: 50000 interactions.

Some questions

- 1 Prove mixing.
- 2 Estimate on the mixing rate?
- 3 Random interaction time \Rightarrow convergence is better?
- 4 Non-equilibrium situation?