Inverse scattering at fixed energy in de Sitter-Reissner-Nordström black holes

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Outline of the talk

- De Sitter-Reissner-Nordström black holes:
  - Spacetime $(\mathcal{M}, g)$ with g lorentzian metric.
  - g completely characterized by M (mass), Q (electric charge) and Λ (cosmological constant).

"Can we determine g by observing waves at infinities?"

- Massless Dirac fields in dS-RN black holes:
  - Dirac waves: $i\partial_t u = \mathbb{D}u$, $\mathbb{D}$ Dirac operator.
  - Scattering matrix $S_g(\lambda)$, λ a fixed energy.
  - Spherical symmetry: $S_g(\lambda) \sim (S_g(\lambda, l))$ where $l \in \frac{1}{2} + \mathbb{N}$ angular momentum.
  - Stationary representation of $S_g(\lambda, l)$ in terms of Jost functions.

- Main result:
  - $g \longrightarrow S_g(\lambda)$ is one-to-one for a fixed energy $\lambda \neq 0$.
  - Proof: complexification of the angular momentum l

De Sitter-Reissner-Nordström Black Holes

- Spacetime $(\mathcal{M}, g)$ with

$$\mathcal{M} = \mathbb{R}_t \times \Sigma, \quad \Sigma =]r_-, r_+[_r \times S^2_{\theta,\varphi},$$

equipped with a Lorentzian metric (signature $(1, -1, -1, -1)$)

$$g = F(r)\, dt^2 - F(r)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2),$$

where

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2.$$

- Singularities: if $Q^2 < \frac{9}{8} M^2$ and $\Lambda M^2 << 1$, then F possesses 3 positive roots: $0 < r_c < r_- < r_+$.

$$\boxed{F(r) > 0 \text{ in } \{r_- < r < r_+\}}$$

$\{r = r_-\}$: event horizon (coordinate singularity).
$\{r = r_+\}$: cosmological horizon (coordinate singularity).

Properties of De Sitter-Reissner-Nordström black holes

- Symmetries:
  - The generators of $SO(3)$ are Killing.
  - The generator of time-translation, ∂_t, is Killing.
- ∂_t timelike, $\partial_r, \partial_\theta, \partial_\varphi$ spacelike.
- $\mathcal{M}$ is static: ∂_t globally defined timelike Killing field, $\perp_g$ to $\Sigma_t = \{t\} \times \Sigma$ everywhere.
- Static observers: integral curves of $U = \frac{1}{\sqrt{F}}\partial_t$. Far from $\{r = r_\pm\}$,

$$t \propto \tau, \quad \tau \text{ proper time.}$$

- Horizons: the null radial geodesics do not reach $\{r = r_\pm\}$ in a finite time t. The horizons are perceived as asymptotic regions by static observers.

Regge-Wheeler variable

- Regge-Wheeler variable: $\frac{dx}{dr} = F^{-1}(r)$

$$x = \frac{1}{2\kappa_n} \ln(r - r_n) + \frac{1}{2\kappa_c} \ln(r - r_c) + \frac{1}{2\kappa_-} \ln(r - r_-) + \frac{1}{2\kappa_+} \ln(r_+ - r) + c,$$

  - $r_n < 0 < r_c < r_- < r_+$: roots of $F(r)$,
  - $\kappa_j = \frac{1}{2} F'(r_j)$, $j = n, c, -, +$ and $\kappa_\pm$ surface gravities of $\{r = r_\pm\}$,
  - c any constant of integration.

- Framework: $\mathcal{M} = \mathbb{R}_t \times \Sigma, \quad \Sigma = \mathbb{R}_x \times S^2_{\theta,\varphi}$,

$$g = F(r)(dt^2 - dx^2) - r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

  - $\{r = r_\pm\} \longleftrightarrow \{x = \pm\infty\}$.
  - Null radial geodesics: $t \pm x = \text{cst}$.

Massless Dirac fields in dS-RN black holes

Hamiltonian form: $i\partial_t \psi = \mathbb{D}\psi, \quad \mathbb{D} = \Gamma^1 D_x + a(x)\mathbb{D}_{S^2}$.

- $\psi \in \mathcal{H} = L^2(\mathbb{R}_x \times S^2, dxd\omega; \mathbb{C}^2)$,
- $D_x = -i\partial_x$,
- Dirac operator on S^2: $\mathbb{D}_{S^2} = -i\Gamma^2(\partial_\theta + \frac{\cot\theta}{2}) - \frac{i}{\sin\theta}\Gamma^3\partial_\varphi$,
- Dirac matrices: $\Gamma^i\Gamma^j + \Gamma^j\Gamma^i = 2\delta_{ij}\mathrm{Id}, \quad \forall i, j = 1, 2, 3.$

$$\Gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

- Potential: $a(x) = \frac{\sqrt{F(r(x))}}{r(x)}$.

$$a(x) = a_\pm e^{\kappa_\pm x} + O(e^{3\kappa_\pm x}), \; x \to \pm\infty,$$

$$a'(x) = a_\pm \kappa_\pm e^{\kappa_\pm x} + O(e^{3\kappa_\pm x}), \; x \to \pm\infty.$$

Spin-weighted spherical harmonics

Spin-weighted spherical harmonics: $L^2(S^2, d\omega; \mathbb{C}^2) = \oplus_{(l,m)\in\mathcal{L}} F_{lm}$,

$$F_{lm}(\theta,\varphi) = \begin{pmatrix} Y^l_{-\frac{1}{2},m}(\theta,\varphi) \\ Y^l_{\frac{1}{2},m}(\theta,\varphi) \end{pmatrix}, \quad (l,m) \in \mathcal{L} = \{l = \frac{1}{2} + \mathbb{N},\ l - |m| \in \mathbb{N}\}.$$

Simplifications

- $\mathcal{H} = \oplus \mathcal{H}_{lm}, \quad \mathcal{H}_{lm} = L^2(\mathbb{R}_x, dx; \mathbb{C}^2) \otimes F_{lm} \simeq L^2(\mathbb{R}_x, dx; \mathbb{C}^2)$,
- $\mathcal{H}_{lm}$ are let invariant through the action of $\mathbb{D}$,
- $\mathbb{D}_{|\mathcal{H}_{lm}} = \mathbb{D}_{lm}, \quad \mathbb{D}_{lm} = \Gamma^1 D_x - (l + \frac{1}{2}) a(x) \Gamma^2$.

Notations:

- $n = l + \frac{1}{2} \in \mathbb{N}^*$ angular momentum.
- $\mathbb{D}_n = \Gamma^1 D_x - n a(x) \Gamma^2$.

Spectral results

Hamiltonians of reference:

- $\mathbb{D}_0 = \Gamma^1 D_x$,
- $\mathbb{D}_n = \Gamma^1 D_x - na(x)\Gamma^2$.

Lemma (Selfadjointness, spectrum)

$(\mathbb{D}_n, \mathbb{D}_0)$ *selfadjoint on* $\mathcal{H} = L^2(\mathbb{R}_x, dx; \mathbb{C}^2)$ *with domains* $H^1(\mathbb{R}_x; \mathbb{C}^2)$. *For all* $n \in \mathbb{N}$,

$$\sigma(\mathbb{D}_n) = \sigma_{ac}(\mathbb{D}_n) = \mathbb{R}.$$

Lemma (Wave and scattering operators)

For all $n \in \mathbb{N}^*$, *the wave operators* $W_n^{\pm} = s - \lim_{t \to \pm\infty} e^{it\mathbb{D}_n} e^{-it\mathbb{D}_0}$ *exist and are asymptotically complete on* $\mathcal{H}$. *The scattering operators* $S_n = (W_n^+)^{-1} W_n^-$ *are unitary on* $\mathcal{H}$ *and commute with* $\mathbb{D}_0$.

The scattering matrix

- Scattering operator: at late times, a solution $e^{-it\mathbb{D}_n}\psi$ can be compared with simpler solutions $e^{-it\mathbb{D}_0}\psi^{\pm}$, $t \to \pm\infty$:

$$\psi^{\pm} = \begin{pmatrix} \psi_1^{\pm} \\ \psi_2^{\pm} \end{pmatrix}, \quad e^{-it\mathbb{D}_0}\psi^{\pm} = \begin{pmatrix} \psi_1^{\pm}(x-t) \\ \psi_2^{\pm}(x+t) \end{pmatrix}.$$

$$S_n : \psi^- \longrightarrow \psi^+.$$

- Scattering matrix: Let $(F_0\psi)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\Gamma^1 \lambda x}\psi(x)dx$. The scattering matrix is defined by

$$S(\lambda, n)\psi(\lambda) = (F_0 S_n F_0^{-1}\psi)(\lambda).$$

For all $n \in \mathbb{N}^*$, $S(\lambda, n)$ is a 2×2-unitary matrix.

Stationary representation of $S(\lambda, n)$

- Stationary equation:

$$[\Gamma^1 D_x - na(x)\Gamma^2]\psi = \lambda\psi, \quad \lambda \in \mathbb{R},\ n \in \mathbb{N}^*. \tag{1}$$

- Jost functions: $F_L(x,\lambda,n)$ and $F_R(x,\lambda,n)$ are the 2×2 matrix-valued solutions of (1) satisfying the asymptotics:

$$\begin{aligned} F_L(x,\lambda,n) &= e^{i\Gamma^1\lambda x}(I_2 + o(1)),\ x \to +\infty, \\ F_R(x,\lambda,n) &= e^{i\Gamma^1\lambda x}(I_2 + o(1)),\ x \to -\infty. \end{aligned}$$

  - F_L and F_R are fundamental matrices of (1): $\det F_{L/R} = 1$.
  - $\exists\ A_L(\lambda,n) = \begin{bmatrix} a_{L1}(\lambda,n) & a_{L2}(\lambda,n) \\ a_{L3}(\lambda,n) & a_{L4}(\lambda,n) \end{bmatrix}$ such that $F_R(\lambda,n) = F_L(\lambda,n)A_L(\lambda,n)$.

- Scattering matrix: $S(\lambda,n) = \begin{bmatrix} T(\lambda,n) & R(\lambda,n) \\ L(\lambda,n) & T(\lambda,n) \end{bmatrix}$, unitary,

$T(\lambda,n) = a_{L1}^{-1}(\lambda,n), \quad R(\lambda,n) = -\frac{a_{L2}(\lambda,n)}{a_{L1}(\lambda,n)}, \quad L(\lambda,n) = \frac{a_{L3}(\lambda,n)}{a_{L1}(\lambda,n)}.$

Dependence on the choice of Regge-Wheeler variable

The Regge-Wheeler variable is defined up to a constant of integration. But x and $\tilde{x} = x + c$ define the same black hole.

The above definition of the scattering matrix depends on the Regge-Wheler variable, *i.e.* on the choice of the constant of integration.

The potential $\tilde{a}$ corresponding to the choice of $\tilde{x}$ satisfies: $\tilde{a}(\tilde{x}) = a(\tilde{x} - c)$.

$$S(\lambda, n) = e^{i\Gamma^1 \lambda c} \tilde{S}(\lambda, n) e^{-i\Gamma^1 \lambda c},$$

Written in components, we have

$$\begin{bmatrix} T(\lambda, n) & R(\lambda, n) \\ L(\lambda, n) & T(\lambda, n) \end{bmatrix} = \begin{bmatrix} \tilde{T}(\lambda, n) & e^{2i\lambda c} \tilde{R}(\lambda, n) \\ e^{-2i\lambda c} \tilde{L}(\lambda, n) & \tilde{T}(\lambda, n) \end{bmatrix}.$$

The main result

Theorem (Daude, Nicoleau, AHP (2010))

Let (M, Q, Λ) and $(\tilde{M}, \tilde{Q}, \tilde{\Lambda})$ be the parameters of two dS-RN black holes. We denote by $a(x)$ and $\tilde{a}(x)$ the corresponding potentials appearing in (1). Let $S(\lambda, n)$ and $\tilde{S}(\lambda, n)$ be the corresponding scattering matrices at a fixed energy $\lambda \neq 0$. Consider a subset $\mathcal{L}$ of $\mathbb{N}^*$ that satisfies the Müntz condition $\sum_{n \in \mathcal{L}} \frac{1}{n} = \infty$ and assume that there exists a constant $c \in \mathbb{R}$ such that one of the following conditions holds:

$$
\begin{array}{rl}
(i) & T(\lambda, n) = \tilde{T}(\lambda, n), \quad \forall n \in \mathcal{L}, \\
(ii) & L(\lambda, n) = e^{-2i\lambda c} \tilde{L}(\lambda, n), \quad \forall n \in \mathcal{L}, \\
(iii) & R(\lambda, n) = e^{2i\lambda c} \tilde{R}(\lambda, n), \quad \forall n \in \mathcal{L}.
\end{array}
$$

Then the potentials a and $\tilde{a}$ coincide up to translations, i.e. there exists a constant $\sigma \in \mathbb{R}$ such that $a(x) = \tilde{a}(x + \sigma)$, $\forall x \in \mathbb{R}$. As a consequence, $M = \tilde{M}$, $Q^2 = \tilde{Q}^2$, $\Lambda = \tilde{\Lambda}$.

Comments

- Zero energy: When $\lambda = 0$,

$$S(0,n) = \begin{pmatrix} \frac{1}{\cosh(An)} & i\tanh(An) \\ i\tanh(An) & \frac{1}{\cosh(An)} \end{pmatrix}, \quad \forall n \in \mathbb{N}^*,$$

  where $A = \int_{\mathbb{R}} a(x)dx$.

- Re-interpretation: Let $\Sigma = \mathbb{R}_x \times S^2_{\theta,\varphi}$ equipped with the Riemanniann metric $g_0 = dx^2 + a^{-2}(x)(d\theta^2 + \sin^2\theta d\varphi^2)$ where $a \in C^\infty(\mathbb{R})$, $a > 0$ and satisfying the asymptotics

$$\begin{aligned} a(x) &= a_\pm e^{\kappa_\pm x} + O(e^{3\kappa_\pm x}), \quad x \to \pm\infty, \\ a'(x) &= a_\pm \kappa_\pm e^{\kappa_\pm x} + O(e^{3\kappa_\pm x}), \quad x \to \pm\infty, \end{aligned}$$

  for some constants $a_\pm > 0$ and $\kappa_+ < 0$, $\kappa_- > 0$. We can bring the Dirac equation on Σ under the same form as above, *i.e.*

$$i\partial_t u = \mathbb{D}_{g_0} u, \quad \mathbb{D}_{g_o} = \Gamma^1 D_x + a(x) D_{S^2}.$$

Main result revisited

Theorem (Daudé, Nicoleau, AHP (2010))

Let $\Sigma = \mathbb{R}_x \times S^2_{\theta,\varphi}$ equipped with $g_0 = dx^2 + a^{-2}(x)(d\theta^2 + \sin^2\theta d\varphi^2)$ satisfying the previous assumptions. Consider the evolution equation $i\partial_t u = \mathbb{D}_{g_0} u$ and the corresponding scattering matrices $S(\lambda, n)$ for all $n \in \mathbb{N}^*$. Then the function $a(x)$ (and thus the metric g_0) is uniquely determined up to translations from the knowledge of either $T(\lambda, n)$ or $R(\lambda, n)$ or $L(\lambda, n)$ for a fixed $\lambda \neq 0$ and for all $n \in \mathcal{L} \subset \mathbb{N}^*$ satisfying the Muntz condition $\sum_{n\in\mathcal{L}} \frac{1}{n} = \infty$.

References

- Asymptotically hyperbolic manifolds (without symmetry assumptions):
  - Joshi, Sa Barreto, Acta Math. [2000]: "asymptotics of the metric uniquely determined from the knowledge of $S(\lambda)$ (associated to the laplacian) at a fixed energy $\lambda \in \mathbb{R}^+$ outside a discret set".
  - Sa Barreto, Duke Math. J. [2005]: "metric uniquely determined from the knowledge of $S(\lambda)$ (associated to the laplacian) for all $\lambda \in \mathbb{R}^+$ except on a discret set of energies".

- De Sitter-Reissner-Nordström black holes:
  - Daude, Nicoleau, Rev. Math. Phys. [2010]: "M, Q^2, Λ uniquely determined by $S(\lambda)$ (associated to Dirac fields) for all $\lambda \in I$, I any open interval of $\mathbb{R}$".

Ideas of the proof

- Complexification of the angular momentum ($a \in L^1(\mathbb{R})$ is enough).
  - Unphysical scattering matrix: $S(\lambda, z)$, $\lambda \in \mathbb{R}$, $z \in \mathbb{C} \setminus \{\text{poles}\}$.
  - Nevanlinna class: $\frac{1}{T(\lambda,z)}, \frac{R(\lambda,z)}{T(\lambda,z)}, \frac{L(\lambda,z)}{T(\lambda,z)} \in N(\Pi^+)$.
  - $S(\lambda, z)$ is uniquely determined on $\mathbb{C} \setminus \{\text{poles}\}$.

- New radial variable: $X = \int_{-\infty}^{x} a(s)ds, \quad X \in (0, A)$.
  - The components $f_{Lj}(X)$ of the Jost function $F_L(X)$ satisfy a non-selfadjoint Sturm-Liouville equation:

$$(L): \qquad u'' + q(X, \lambda)u = z^2 u$$

$$q(X, \lambda) = \frac{\omega^-}{X^2} + O(1), \; X \to 0,$$

$$q(X, \lambda) = \frac{\omega^+}{(A-X)^2} + O(1), \; X \to A.$$

  - $iza_{L1}(\lambda, z) = W(f_{L1}(X), f_{R2}(X))$, $iza_{L2}(\lambda, z) = W(f_{L2}(X), f_{R2}(X))$, etc. . . .
  - Asymptotics of $a_{Lj}(\lambda, z)$ and of the scattering coefficients when $z \to \infty$.
  - Inverse result by classical method.

Complexification of the angular momentum

- Stationary equation:

$$[\Gamma^1 D_x - za(x)\Gamma^2]\psi = \lambda\psi, \quad \lambda \in \mathbb{R},\ z \in \mathbb{C}. \qquad (2)$$

- Jost functions $F_L(x,\lambda,z)$ and $F_R(x,\lambda,z)$: solutions of (2) satisfying the integral equations:

$$\begin{aligned} F_L(x,\lambda,z) &= e^{i\Gamma^1\lambda x} - iz\Gamma^1 \int_x^{+\infty} e^{-i\Gamma^1\lambda(y-x)} a(y)\Gamma^2 F_L(y,\lambda,z)dy, \\ F_R(x,\lambda,z) &= e^{i\Gamma^1\lambda x} + iz\Gamma^1 \int_{-\infty}^{x} e^{-i\Gamma^1\lambda(y-x)} a(y)\Gamma^2 F_R(y,\lambda,z)dy. \end{aligned}$$

  - solvable by an iterative procedure.
  - $z \longrightarrow F_L(x,\lambda,z),\ F_R(x,\lambda,z)$ are analytic on $\mathbb{C}$.

Complexification of the angular momentum

Scattering data A_L:

$$A_L(\lambda, z) = I_2 - iz\Gamma^1 \int_{\mathbb{R}} e^{-i\Gamma^1 \lambda y} a(y) \Gamma^2 F_L(y, \lambda, z) dy$$

extends analytically to $\mathbb{C}$ as well.

Lemma

- $a_{Lj} \in H(\mathbb{C}), \quad |a_{Lj}(\lambda, z)| \leq e^{A|z|}, \quad A = \int_{\mathbb{R}} a(x)dx.$
- $z \longrightarrow a_{L1}(\lambda, z), a_{L4}(\lambda, z)$ *even*, $\quad z \longrightarrow a_{L2}(\lambda, z), a_{L3}(\lambda, z)$ *odd.*

Remark: When $\lambda = 0$, explicit calculations can be made:
$a_{L1}(0, z) = a_{L4}(0, z) = \cosh(zA), \quad a_{L2}(0, z) = -a_{L3}(0, z) = -i \sinh(zA).$

Complexification of the angular momentum

Lemma

$$|a_{Lj}(\lambda, z)| \leq e^{A|Re(z)|}.$$

Proof: The unitarity of $S(\lambda, z)$ for $z \in \mathbb{R}$ extends analytically to

$$a_{L1}(\lambda, z)\overline{a_{L1}(\lambda, \bar{z})} - a_{L3}(\lambda, z)\overline{a_{L3}(\lambda, \bar{z})} = 1, \ \forall z \ \in \mathbb{C}.$$

- Imaginary axis: $z = it, \ t \in \mathbb{R}, \ |a_{L1}(\lambda, it)|^2 + |a_{L3}(\lambda, it)|^2 = 1.$ Hence

$$|a_{Lj}(\lambda, it)| \leq 1, \ \forall t \in \mathbb{R}.$$

- Real axis: $|a_{Lj}(\lambda, x)| \leq e^{A|x|}, \ \forall x \in \mathbb{R}.$
- By the Phragmen-Lindelöf Thm, $|a_{Lj}(\lambda, z)| \leq e^{A|Re(z)|}.$

Nevanlinna class

Theorem (Nevanlinna class, Uniqueness)

A function f belongs to $N(\Pi^+)$, where $\Pi^+ = \{z \in \mathbb{C} : \ Re(z) > 0\}$, if it is analytic on Π^+ and if

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \ln^+ \left| f\left(\frac{1 - re^{i\varphi}}{1 + re^{i\varphi}}\right)\right| d\varphi < \infty,$$

where $\ln^+(x) = \begin{cases} \ln x, & \ln x \geq 0, \\ 0, & \ln x < 0. \end{cases}$

If $f \in N(\Pi^+)$ satisfies $f(n) = 0$ for all $n \in \mathcal{L} \subset \mathbb{N}^*$ such that $\displaystyle\sum_{n \in \mathcal{L}} \frac{1}{n} = \infty$,

then $f \equiv 0$ in Π^+.

Nevanlinna class and applications

Lemma

- $a_{Lj}(\lambda, z)_{|\Pi^+} \in N(\Pi^+)$.
- *If* $a_{Lj}(\lambda, n) = \tilde{a}_{Lj}(\lambda, n), \forall n \in \mathcal{L}$, *then*

$$a_{Lj}(\lambda, z) = \tilde{a}_{Lj}(\lambda, z), \forall z \in \mathbb{C}.$$

Proof: Assume $|f(z)| \leq e^{ARe(z)}, \ \forall z \in \Pi^+$.

$$\log^+ \left| f\Big(\frac{1 - re^{i\varphi}}{1 + re^{i\varphi}}\Big)\right| \leq A.Re\Big(\frac{1 - re^{i\varphi}}{1 + re^{i\varphi}}\Big) \leq \frac{1 - r^2}{1 + r^2 + 2r\cos\varphi}.$$

We conclude using $\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2+2r\cos\varphi} d\varphi = 2\pi$.

Nevanlinna class and applications

Lemma

Assume there exists $c \in \mathbb{R}$ such that $L(\lambda, n) = e^{-2i\lambda c}\tilde{L}(\lambda, n), \ \forall n \in \mathcal{L}$. Then, for all $z \in \mathbb{C}$,

$$a_{L1}(\lambda, z) = \tilde{a}_{L1}(\lambda, z), \quad a_{L2}(\lambda, z) = e^{2i\lambda c}\tilde{a}_{L2}(\lambda, z),$$
$$a_{L3}(\lambda, z) = e^{-2i\lambda c}\tilde{a}_{L3}(\lambda, z), \quad a_{L4}(\lambda, z) = \tilde{a}_{L4}(\lambda, z).$$

Proof:

- By assumption, $a_{L3}(\lambda, n)\tilde{a}_{L1}(\lambda, n) = e^{-2i\lambda c}a_{L1}(\lambda, n)\tilde{a}_{L3}(\lambda, n), \ \forall n \in \mathcal{L}$. But $z \longrightarrow a_{L3}(\lambda, z)\tilde{a}_{L1}(\lambda, z) \in N$. Thus

$$a_{L3}(\lambda, z)\tilde{a}_{L1}(\lambda, z) = e^{-2i\lambda c}a_{L1}(\lambda, z)\tilde{a}_{L3}(\lambda, z), \quad \forall z \in \mathbb{C}. \qquad (3)$$

- (Unitarity) $a_{L1}(\lambda, z)\overline{a_{L1}(\lambda, \bar{z})} = 1 - a_{L3}(\lambda, z)\overline{a_{L3}(\lambda, \bar{z})}, \ \forall z \in \mathbb{C}$. Thus a_{L1} and a_{L3} have no common zeros and (3) entails that a_{Lj} and $\tilde{a}_{Lj}$ have the same zeros with the same multiplicity.

Nevanlinna class and applications

- Recall that $a \longrightarrow a_{L1}(\lambda, z)$ even and $a_{L1}(\lambda, 0) = 1$. Hence $a_{L1}(\lambda, z) = f(z^2)$ where $f(z) \in H(\mathbb{C})$ and is of order $\frac{1}{2}$. By Hadamard's factorization Thm, we get:

$$a_{L1}(\lambda, z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_n^2}\right) = \tilde{a}_{L1}(\lambda, z).$$

Remark: The Lemma remains true if we assume one of the following

- $T(\lambda, n) = \tilde{T}(\lambda, n), \quad \forall n \in \mathcal{L},$
- $R(\lambda, n) = e^{2i\lambda c}\tilde{R}(\lambda, n), \quad \forall n \in \mathcal{L}.$

A first inverse result

Corollary (First inverse result)

We assume that the hypotheses of the main Thm are satisfied for all λ in an open interval I. Then the potential $a(x)$ is uniquely determined up to translations.

Proof:

- By the previous Lemma,

$$a_{L2}(\lambda, z) = e^{2i\lambda c}\tilde{a}_{L2}(\lambda, z), \quad \forall z \in \mathbb{C}, \ \forall \lambda \in I.$$

- We let $z \to 0$ and we look at the first term in the entire serie defining a_{L2}. Then

$$\hat{a}(2\lambda) = e^{2i\lambda c}\hat{\tilde{a}}(2\lambda), \quad \forall \lambda \in I. \tag{4}$$

- $a(x)$ is exponentially decreasing. Hence, the equality (4) extends analytically to a small strip containing $\mathbb{R}$. Hence, $a(x) = \tilde{a}(x - c), \ \forall x \in \mathbb{R}$.

A new radial variable

- Radial variable X: Set $A = \int_{\mathbb{R}} a(x)dx$. Define

$$g: \quad \mathbb{R} \longrightarrow (0, A), \\ x \longrightarrow g(x) = X = \int_{-\infty}^{x} a(s)ds.$$

- Notations: $f_{Lj/Rj}(X, \lambda, z) := f_{Lj/Rj}(g^{-1}(X), \lambda, z)$.
- The components $f_{L1}(X, \lambda, z), f_{L2}(X, \lambda, z), f_{R1}(X, \lambda, z), f_{R2}(X, \lambda, z)$ satisfy a non-selfadjoint singular Sturm-Liouville equation:

$$\left[\frac{d^2}{dX^2} + q(X, \lambda)\right] f(X'\lambda, z) = z^2 f(X, \lambda, z),$$

$$q(X, \lambda) = \frac{\lambda^2}{a^2(x)} + i\lambda \frac{a'(x)}{a^3(x)}.$$

$$q(X, \lambda) = \frac{\omega^-}{X^2} + O(1), \ X \to 0, \quad \omega_- = \frac{\lambda^2}{\kappa_-^2} + i\frac{\lambda}{\kappa_-},$$

$$q(X, \lambda) = \frac{\omega^+}{(A-X)^2} + O(1), \ X \to A, \quad \omega_+ = \frac{\lambda^2}{\kappa_+^2} + i\frac{\lambda}{\kappa_+}.$$

Asymptotics of the a_{Lj}

- Remark: The spectral data a_{Lj} are interpreted in terms of Wronskians of the $f_{Lj/Rj}(X)$, *i.e.*

$$a_{L1}(\lambda, z) = \frac{1}{iz} W(f_{L1}(X), f_{R2}(X)), a_{L2}(\lambda, z) = \frac{1}{iz} W(f_{L2}(X), f_{R2}(X)),$$
$$a_{L3}(\lambda, z) = -\frac{1}{iz} W(f_{L1}(X), f_{R1}(X)), a_{L4}(\lambda, z) = -\frac{1}{iz} W(f_{L2}(X), f_{R1}(X))$$

- Asymptotics: The components $f_{L1}(X, \lambda, z), f_{L2}(X, \lambda, z)$, $f_{R1}(X, \lambda, z), f_{R2}(X, \lambda, z)$ are perturbations of modified Bessel functions whose asymptotics are well-known. Using the previous Wronskians, we get for example:

$$a_{L4}(\lambda, z) \sim Const.z^{i\lambda\left(\frac{1}{\kappa_+} - \frac{1}{\kappa_-}\right)} e^{zA}, \quad z \longrightarrow +\infty.$$

- An application: Since the a_{Lj} are uniquely determined, we get:

$$\int_{\mathbb{R}} a(x)dx = A = \tilde{A} = \int_{\mathbb{R}} \tilde{a}(x)dx.$$

Application to the inverse problem

Consider two black holes with parameters (M, Q^2, Λ) and $(\tilde{M}, \tilde{Q}, \tilde{\Lambda})$. Denote by $a(x)$ and $\tilde{a}(x)$ the corresponding potentials. We assume that

$$\begin{array}{rl} (i) & T(\lambda, n) = \tilde{T}(\lambda, n), \quad \forall n \in \mathcal{L}, \\ (ii) & L(\lambda, n) = e^{-2i\lambda c}\tilde{L}(\lambda, n), \quad \forall n \in \mathcal{L}, \\ (iii) & R(\lambda, n) = e^{2i\lambda c}\tilde{R}(\lambda, n), \quad \forall n \in \mathcal{L}. \end{array}$$

- For simplicity, take $c = 0$ in the above formulae.
- Introduce

$$g: \begin{array}{l} \mathbb{R} \longrightarrow (0, A), \\ x \longrightarrow g(x) = \int_{-\infty}^{x} a(s)ds. \end{array}, \quad \tilde{g}: \begin{array}{l} \mathbb{R} \longrightarrow (0, \tilde{A}), \\ x \longrightarrow \tilde{g}(x) = \int_{-\infty}^{x} \tilde{a}(s)ds. \end{array}.$$

- Since $A = \tilde{A}$, set $X \in (0, A)$ and

$$F_R(X, \lambda, z) = F_R(g^{-1}(X), \lambda, z), \quad \tilde{F_R}(X, \lambda, z) = \tilde{F_R}(\tilde{g}^{-1}(X), \lambda, z).$$

Application to the inverse problem

- For $X \in (0, A)$, define the 2×2 matrix-valued function $P(X, \lambda, z)$ by
$$P(X,\lambda,z)\tilde{F_R}(X,\lambda,z) = F_R(X,\lambda,z).$$
- By inverting $\tilde{F_R}$, we get for instance
$$P_1(X,\lambda,z) = 1 + \frac{F_{R1}(X)}{a_{L4}}(\tilde{F}_{L4}(X) - F_{L4}(X)) + \frac{F_{L2}(X)}{a_{L4}}(F_{R3}(X) - \tilde{F}_{R3}(X)),$$
$$P_2(X,\lambda,z) = \frac{1}{a_{L4}}(F_{L2}(X)\tilde{F}_{R1}(X) - \tilde{F}_{L2}(X)F_{R1}(X)).$$
- From the properties of $F_R(X, \lambda, z)$, it is easy to show that the functions $z \longrightarrow P_j(X, \lambda, z)$ belong to $H(\mathbb{C})$, are of exponential type and are bounded on $i\mathbb{R}$.
- Using simple algebraic manipulations and the previous asymptotics on the $a_{Lj}(\lambda, z)$, $z \to \infty$, we can show that $z \longrightarrow P_j(X, \lambda, z)$ are also bounded on $\mathbb{R}$.

Application to the inverse problem

- Phragmen-Lindelöf Thm: $z \longrightarrow P_j(X, \lambda, z)$ are bounded on $\mathbb{C}$.
- Liouville Thm: $P_j(X, \lambda, z) = P_j(X, \lambda, 0)$ for all $z \in \mathbb{C}$.
- In the case $z = 0$, explicit calculations lead to

$$P_j(X, \lambda, z) = e^{i\Gamma^1 \lambda (g^{-1}(X) - \tilde{g}^{-1}(X))}, \ \forall z \in \mathbb{C}.$$

- From the definition $P(X, \lambda, z)\tilde{F}_R(X, \lambda, z) = F_R(X, \lambda, z)$, we thus obtain

$$\tilde{F}_R(X, \lambda, z) = e^{i\lambda\Gamma^1 (\tilde{g}^{-1}(X) - g^{-1}(X))} F_R(X, \lambda, z), \ j = 1, 2.$$

In particular,

$$\tilde{f}_{Rj}(X, \lambda, z) = e^{i\lambda (\tilde{g}^{-1}(X) - g^{-1}(X))} f_{Rj}(X, \lambda, z), \ j = 1, 2.$$

Application to the inverse problem

- Using that the Wronskians

$$W(f_{R1}(X,\lambda,z), F_{R2}(X,\lambda,z)) = iz,$$
$$W(\tilde{F}_{R1}(X,\lambda,z), \tilde{F}_{R2}(X,\lambda,z)) = iz,$$

we obtain

$$e^{i\lambda(\tilde{g}^{-1}(X)-g^{-1}(X))} = 1.$$

Hence, by a continuity argument

$$\tilde{g}^{-1}(X) = g^{-1}(X) + \frac{k\pi}{\lambda}, \tag{5}$$

- Taking the derivative of (5) w.r.t X, we get:

$$\frac{1}{\tilde{a}(\tilde{g}^{-1}(X))} = \frac{1}{a(g^{-1}(X))}, \quad \forall X \in (0, A).$$

Equivalently, $a(x) = \tilde{a}(x + \frac{k\pi}{\lambda})$.