Localizations at infinity of operators and applications

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Introduction/1

- H = self-adjoint operator on $\mathcal{H} \Rightarrow$ the localization at infinity of H is $\boxed{\widehat{H} = H/K(\mathcal{H})}$
  - An "abstract operator" which describes the behavior at infinity of H.
  - Observable affiliated to the Calkin algebra $C(\mathcal{H}) := L(\mathcal{H})/K(\mathcal{H})$ (which is a monster).
  - Operation which does not make sense at a purely Hilbertian level.
  - The simplest thing one can do with it: compute the essential spectrum of H: $\boxed{\mathrm{Sp}_{\mathrm{ess}}(H) = \mathrm{Sp}(\widehat{H})}$
  - More: prove the Mourre estimate for H w.r.t. a second self-adjoint operator A.
- The main difficulty is to obtain a convenient (explicit) representation of the "abstract" object $\widehat{H}$ i.e. to construct a new Hilbert space $\widehat{\mathcal{H}}$ and a realization of $\widehat{H}$ as a self-adjoint operator on it.
- The idea is to try to find a C^*-algebra $\mathscr{C} \subset L(\mathcal{H})$ such that

$$\boxed{(1)\ H \in' \mathscr{C} \quad (2)\ K(\mathcal{H}) \subset \mathscr{C} \quad (3)\ \widehat{\mathscr{C}} := \mathscr{C}/K(\mathcal{H}) \text{ is explicitly realized on some Hilbert } \widehat{\mathcal{H}}}$$

- Think about representing a region of a complicated topological manifold with the help of a coordinate chart.

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- Think about representing a region of a complicated topological manifold with the help of a coordinate chart.

Introduction/2

- **What can go wrong if $\mathscr{C}$ is not well chosen:** Let H without eigenvalues and
  - $C^*(H) = C^*$-algebra generated by H
  - $\mathscr{C} := K(\mathcal{H}) + C^*(H) =$ smallest C^*-algebra to which H is affiliated and contains the compacts
  - the sum $\mathscr{C} = K(\mathcal{H}) + C^*(H)$ is direct hence $\mathscr{C}/K(\mathcal{H}) = C^*(H)$ hence $\widehat{\mathcal{H}} = \mathcal{H}$ and $\widehat{H} = H$ so $\mathrm{Sp}_{\mathrm{ess}}(H) = \mathrm{Sp}(H)$ which is true but trivial.

- **Example of good choice: in the one dimensional anisotropic case**
  - We are interested in $H = P^2 + V(x)$ with $\lim_{x\to\pm\infty} V(x) = V_\pm$
  - We take $\mathscr{C} := \mathcal{C}(\overline{\mathbb{R}}) \cdot C^*(\mathbb{R})$
  - We get $\widehat{H} = (P^2 + V_-) \oplus (P^2 + V_+)$, $\mathrm{Sp}_{\mathrm{ess}}(H) = [\min(V_-, V_+), \infty[$

- In general: instead of studying an operator H, study a C^*-algebra $\mathscr{C}$.

  The choice of $\mathscr{C}$ is determined by the algebraic structure of H: try to find the smallest algebra to which certain "elementary Hamiltonians" are affiliated. At the end use various affiliation criteria to prove that the Hamiltonian which is of interest is affiliated to $\mathscr{C}$.

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Notations, Definitions

- $\mathcal{H}$ = Hilbert space $\Rightarrow$ $L(\mathcal{H})$ = bounded operators, $K(\mathcal{H})$ = compact operators
- $H \in L(\mathcal{H}) \Rightarrow$ $\widehat{H}$ = its image in $C(\mathcal{H}) = L(\mathcal{H})/K(\mathcal{H})$
- $\mathscr{C} \subset L(\mathcal{H})$ C^*-subalgebra with $K(\mathcal{H}) \subset \mathscr{C}$ then $\widehat{\mathscr{C}} := \mathscr{C}/K(\mathcal{H})$ = abstract C^*-algebra
- An unbounded self-adjoint operator is identifed with its C_0 functional calculus. More generally:
- An observable affiliated to a C^*-algebra $\mathscr{C}$ is a morphism $H : \mathcal{C}_\mathrm{o}(\mathbb{R}) \to \mathscr{C}$. Write $H \in' \mathscr{C}$.

  Thus for a usual self-adjoint operator we have to set $H(\theta) \equiv \theta(H)$ for $\theta \in \mathcal{C}_\mathrm{o}(\mathbb{R})$.
- $\mathcal{P} : \mathscr{C} \to \mathscr{C}'$ morphism $\Rightarrow \mathcal{P}(H) := \mathcal{P} \circ H$ observable affiliated to $\mathscr{C}'$.
- A self-adjoint operator H on $\mathcal{H}$ is affiliated to $\mathscr{C}$ if it satisfies

$$(H - z)^{-1} \in \mathscr{C} \text{ for some } z \quad \Big(\Longleftrightarrow \theta(H) \in \mathscr{C} \quad \forall \theta \in C_0(\mathbb{R}) \Big)$$

- H self-adjoint on $\mathcal{H} \Rightarrow C^*(H)$ = smallest C^*-algebra which contains all $\varphi(H)$, $\varphi \in \mathcal{C}_\mathrm{o}(\mathbb{R})$.
- X = f.d.r. vector space: $\mathcal{C}(X) \equiv \mathcal{C}_\mathrm{b}^\mathrm{u}(X)$ = C^*-algebra of bounded uniformly continuous functions
  - $\mathcal{C}_\mathrm{o}(X)$ = continuous functions which tend to zero at infinity $\subset \mathcal{C}(X)$
  - $\mathcal{C}_\infty(X)$ = continuous functions which have a limit at infinity = $\mathcal{C}_\mathrm{o}(X) + \mathbb{C} \subset \mathcal{C}(X)$
  - $Y \subset X$ subspace $\Rightarrow \mathcal{C}_\mathrm{o}(X/Y) \subset \mathcal{C}(X)$ $\big(\varphi \mapsto \varphi \circ \pi_Y$ where $\pi_Y : X \to X/Y$ projection$\big)$
  - $C^*(X)$ = group C^*-algebra = algebra generated by translations $\cong \mathcal{C}_\mathrm{o}(X^*) = \{\varphi(P) \mid \varphi \in \mathcal{C}_\mathrm{o}(X^*)\}$

Hamiltonian and conjugate operator

- $H, A =$ self-adjoint operators on $\mathcal{H}$ with H of class[1] $C^1_{\mathrm{u}}(A)$. Then
  - $\mathcal{A}(H) := [H, iA]$ continuous sesquilinear form on $D(H)$
  - $\theta \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}), \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ and $\psi(x)\theta(x) = x\theta(x) \Rightarrow$

$$\theta(H)^*[H, \mathrm{i}A]\theta(H) = \partial_{\tau}{}_{|_{\tau=0}} \theta(H)^* \mathrm{e}^{-\mathrm{i}\tau A}\psi(H)\mathrm{e}^{\mathrm{i}\tau A}\theta(H)$$

- $\rho^A_H : \mathbb{R} \to]-\infty, +\infty]$ lower semicontinuous, $\boxed{\mathrm{Sp}(H) = \{\rho < \infty\}}$

$$\rho^A_H(\lambda) = \sup\{a \in \mathbb{R} \mid \exists\theta \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}),\ \theta(\lambda) \neq 0,\ \text{such that}\\ \theta(H)^*[H, \mathrm{i}A]\theta(H) \geq a|\theta(H)|^2 \quad \}$$

- $\widehat{\rho}^A_H : \mathbb{R} \to]-\infty, +\infty]$ lower semicontinuous, $\boxed{\mathrm{Sp}_{\mathrm{ess}}(H) = \{\widehat{\rho} < \infty\}}$

$$\widehat{\rho}^A_H(\lambda) = \sup\{a \in \mathbb{R} \mid \exists\theta \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}),\ \theta(\lambda) \neq 0, \exists K = \text{ compact},\\ \theta(H)^*[H, \mathrm{i}A]\theta(H) \geq a|\theta(H)|^2 + K \quad \}$$

[1] $H \in C^1_{\mathrm{u}}(A) \Leftrightarrow$ the map $\tau \mapsto \mathrm{e}^{-i\tau A}(H+i)^{-1}\mathrm{e}^{i\tau A}$ is norm C^1

A C*-algebra framework for the Mourre estimate

- $\mathscr{C}$ a C^*-algebra of operators on $\mathcal{H}$ such that $K(\mathcal{H}) \subset \mathscr{C}$
- $\widehat{\mathscr{C}} = \mathscr{C}/K(\mathcal{H})$ "abstract" C^*-algebra; $\mathcal{P} : \mathscr{C} \to \widehat{\mathscr{C}}$ natural morphism
- $\mathcal{U}_\tau[S] = \mathrm{e}^{\tau\mathcal{A}}[S] = \mathrm{e}^{-\mathrm{i}\tau A} S \mathrm{e}^{\mathrm{i}\tau A}$ is a norm continuous group of automorphisms of $\mathscr{C}$
- $\widehat{\mathcal{U}}_\tau = \mathrm{e}^{\tau\widehat{\mathcal{A}}}$ induced group of automorphisms of $\widehat{\mathscr{C}}$
- *H observable affiliated to* $\mathscr{C}$ of class $C^1_{\mathrm{u}}(\mathcal{A}) \Longrightarrow \widehat{\rho}_H$ is well defined
- $\widehat{H} = \mathcal{P}(H)$ affiliated to $\widehat{\mathscr{C}}$ of class $C^1_{\mathrm{u}}(\widehat{\mathcal{A}}) \Longrightarrow \rho_{\widehat{H}}$ is well defined
- We have $\boxed{\widehat{\rho}_H = \rho_{\widehat{H}}}$

Remark: When $\widehat{\mathscr{C}}$ is represented on $\widehat{\mathcal{H}}$ the group of automorphisms $\widehat{\mathcal{U}}_\tau$ is usually implemented by a unitary group $\widehat{U}_\tau = \mathrm{e}^{\mathrm{i}\tau\widehat{A}}$ which is simpler than the initial $U_\tau = \mathrm{e}^{\mathrm{i}\tau A}$.

Exceptional eigenvalues, critical points, thresholds

Theorem: (ρ and $\widehat{\rho}$ coincide except at A-exceptional eigenvalues)

- $\lambda =$ eigenvalue of H, $\widehat{\rho}^A_H(\lambda) > 0 \Rightarrow \lambda$ has finite multiplicity and H has no other eigenvalues in $[\lambda - \varepsilon, \lambda + \varepsilon]$ for ε small

- let $\epsilon_A(H) = \{\lambda =$ eigenvalue of H $\mid \widehat{\rho}^A_H(\lambda) > 0\}$, then

$$\rho^A_H(\lambda) = \begin{cases} 0 & \text{if } \lambda \in \epsilon_A(H), \\ \widehat{\rho}^A_H(\lambda) & \text{if } \lambda \notin \epsilon_A(H). \end{cases}$$

- Define

$$\varkappa_A(H) = \{\rho^A_H \leq 0\} = \text{(closed) set of } A\text{-}\textbf{critical points} \text{ of } H$$

$$\tau_A(H) = \{\widehat{\rho}^A_H \leq 0\} = \text{(closed) set of } A\text{-}\textbf{thresholds} \text{ of } H$$

Then

$$\varkappa_A(H) = \tau_A(H) \sqcup \epsilon_A(H) = \tau_A(H) \cup \{\text{ eigenvalues of } H\}$$

Computing ρ and $\widehat{\rho}$: elementary facts

- **Theorem (stability under compact perturbation):**

$$H_1, H_2 \in C^1_{\mathrm{u}}(A) \quad \text{and} \quad (H_1 - \mathrm{i})^{-1} - (H_2 - \mathrm{i})^{-1} \in K(\mathcal{H}) \quad \Longrightarrow \widehat{\rho}^{\,A}_{H_1} = \widehat{\rho}^{\,A}_{H_2}$$

- This solves the problem in the "two-body" case: then, given H, there are compact perturbations H_0 of H which are very simple, in particular $\widehat{\rho}^{\,A}_{H_0}$ is explicit.

- **Theorem (direct sums):** $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$

$$H = H_1 \oplus H_2 \text{ and } A = A_1 \oplus A_2 \Rightarrow \rho = \inf(\rho_1, \rho_2)$$

- **Theorem (tensor products):** $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

If $H = H_1 \otimes 1 + 1 \otimes H_2$ and $A = A_1 \otimes 1 + 1 \otimes A_2$ then

$$\rho(\lambda) = \inf_{\lambda = \lambda_1 + \lambda_2} \Big(\rho_1(\lambda_1) + \rho_2(\lambda_2)\Big)$$

C*-algebras of quantum Hamiltonians

- We shall consider four examples
  - One dimensional anisotropic systems (toy model)
  - N-body systems
  - Many-body systems
  - Quantum fields
- In each case we construct $\mathscr{C}$ as the C^*-algebra generated by the operators which are natural Hamiltonians in the respective physical situation.
- We shall see that these algebras are remarkable mathematical objects:
- **Theorem:** X = f.d.r. vector space (configuration space).
  Let $h : X^* \to \mathbb{R}$ (kinetic energy) be continuous and divergent at infinity.
  Let $v : X \to \mathbb{R}$ (potential energy) be bounded and uniformly continuous.
  Let $C^*(v)$ be the C^*-algebra generated by the translates of v.
  Let $W_\xi = e^{i(kQ+xP)}, \quad \xi = (x, k)$.
  The C^*-algebra of operators on $L^2(X)$ generated by the self-adjoint operators
  $$W_\xi^*[h(P) + \lambda v(Q)]W_\xi \equiv h(P-k) + \lambda v(Q+x) \quad \text{with } k \in X^*, x \in X, \lambda \in \mathbb{R}$$
  is the crossed product $\boxed{C^*(v) \rtimes X = C^*(v) \cdot C^*(X)}$

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2-body systems

- **2-body algebra:** algebra which is naturally represented on the same Hilbert space $\mathcal{H}$ as $\mathscr{C}$.

  More precisely: $\mathscr{C} = \mathscr{D} + K(\mathcal{H})$ where $\mathscr{D} \subset L(\mathcal{H})$ is a C^*-algebra with $\mathscr{D} \cap K(\mathcal{H}) = 0$;
  then the sum is topologically direct, $\mathscr{C}$ is a C^*-algebra, and $\widehat{\mathscr{C}} \cong \mathscr{D}$.

- As explained before, this is not a good notion.

- 2-body algebra of a vector space

  - X = f.d.r. vector space, $\mathcal{H} = L^2(X)$, $\quad \mathscr{K}(X) = K(L^2(X))$
  - $\mathcal{C}_\infty(X)$ = continuous functions which have a limit at infinity = $\mathcal{C}_{\mathrm{o}}(X) + \mathbb{C} \subset \mathcal{C}(X)$
  - The associated 2-body algebra:

    $$\begin{aligned}\mathscr{C} := \mathcal{C}_\infty(X) \rtimes X &= \mathcal{C}_\infty(X) \cdot C^*(X) = \mathbb{C} \cdot C^*(X) + \mathcal{C}_{\mathrm{o}}(X) \cdot C^*(X) \\ &= C^*(X) + \mathscr{K}(X)\end{aligned}$$

  - 2-body Hamiltonian: H = self-adjoint operator on $L^2(X)$ such that

    $\exists h : X^* \to \mathbb{R}$ continuous divergent at infinity with $(H+\mathrm{i})^{-1} - (h(P)+\mathrm{i})^{-1}$ = compact

  - (1) $\mathscr{C} = C^*(X) + \mathscr{K}(X)$ on $\mathcal{H} = L^2(X)$; $\quad$ (2) $\widehat{\mathscr{C}} = C^*(X)$ on $\widehat{\mathcal{H}} = \mathcal{H}$; $\quad$ (3) $\widehat{H} = h(P)$
  - $\widehat{\rho}^{A}_{H} = \widehat{\rho}^{A}_{h(P)} = \rho^{A}_{h(P)}$ $\quad$ because $h(P)$ has no eigenvalues of finite multiplicity.
  - Note: $H \in C^1_{\mathrm{u}}(A) \Leftrightarrow [A, (H+\mathrm{i})^{-1}] \in \mathscr{C}$

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  - The associated 2-body algebra:

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One dimensional anisotropic systems

- $\bar{} = [-\infty, +\infty]$, $\quad \mathcal{C}(\bar{\mathbb{R}}) =$ set of continuous functions which have finite limits at $\pm\infty$.
- Compute $\mathcal{C}(\bar{\mathbb{R}})/\mathcal{C}_{\mathrm{o}}(\mathbb{R})$: if $L_\pm(v) := \lim_{x\to\pm\infty} v(x)$ then $L_\pm : \mathcal{C}(\bar{\mathbb{R}}) \to \mathbb{C}$ are morphisms, hence
  $L := (L_-, L_+) : \mathcal{C}(\bar{\mathbb{R}}) \to \mathbb{C} \oplus \mathbb{C}$ is a morphism, and clearly $\ker L = \mathcal{C}_{\mathrm{o}}(\mathbb{R})$. Thus
  $\mathcal{C}(\bar{\mathbb{R}})/\mathcal{C}_{\mathrm{o}}(\mathbb{R}) \cong \mathbb{C} \oplus \mathbb{C}$ isomorphism implemented by the morphism $L : \mathcal{C}(\mathbb{R}) \to \mathbb{C} \oplus \mathbb{C}$
- Hamiltonian algebra: $\mathscr{C} = \mathcal{C}(\bar{\mathbb{R}}) \rtimes \mathbb{R} = \mathcal{C}(\bar{\mathbb{R}}) \cdot C^*(\mathbb{R})$ realized on $\mathcal{H} = L^2(\mathbb{R})$
  - $\mathscr{C} \supset \mathscr{K} = K(L^2(\mathbb{R})) = \mathcal{C}_{\mathrm{o}}(\mathbb{R}) \cdot C^*(\mathbb{R})$
  - $L_\pm$ induce morphisms $\mathcal{P}_\pm : \mathscr{C} \to C^*(\mathbb{R})$ by the rule $\mathcal{P}_\pm[\varphi(Q)\psi(P)] = L_\pm(\varphi)\psi(P)$
  - $\boxed{\widehat{\mathscr{C}} = \mathscr{C}/\mathscr{K} \subset C^*(\mathbb{R}) \oplus C^*(\mathbb{R})}$ implemented by the morphism $\mathcal{P} = (\mathcal{P}_-, \mathcal{P}_+)$
  - $\widehat{\mathscr{C}}$ is realized on $\widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$
- $H \in' \mathscr{C} \Rightarrow \mathcal{P}[H] = H_- \oplus H_+$ with $H_\pm = \mathcal{P}_\pm[H] \in' C^*(\mathbb{R})$.
- $\mathrm{Sp}_{\mathrm{ess}}(H) = \mathrm{Sp}(H_-) \cup \mathrm{Sp}(H_+)$.
- For "reasonable" A we get $\quad \widehat{\rho}_H^A = \inf(\rho_{H_-}^{A_-}, \rho_{H_+}^{A_+}) \quad$ which is quite explicit because $H_\pm$ are functions of momentum

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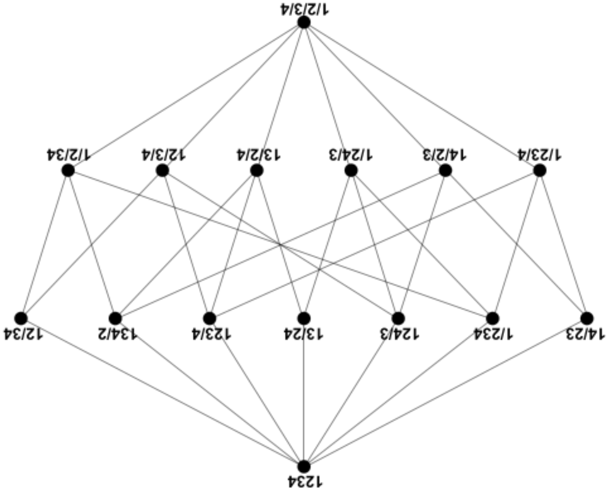
N-body systems (ABM+VG 1991+1993)

- $X =$ real finite dimensional vector space
- $Y \subset X$ linear subspace $\Rightarrow$ natural embedding $\mathcal{C}_o(X/Y) \subset \mathcal{C}_b^u(X) \quad (\varphi \mapsto \varphi \circ \pi_Y)$
- $\mathcal{T} =$ set of subspaces of X
- $\mathcal{S} =$ set of finite intersections of elements of $\mathcal{T}$
- $\mathcal{C}_X(\mathcal{S}) := \sum^{c}_{Y \in \mathcal{S}} \mathcal{C}_0(X/Y) = C^*$-algebra generated by $\bigcup_{Y \in \mathcal{T}} \mathcal{C}_0(X/Y)$
- The C^*-algebra of operators on $L^2(X)$ generated by the operators

$$h(P-k) + v(Q) \quad \text{with } k \in X^*, v \in \sum_{Y \in \mathcal{T}} \mathcal{C}_c^\infty(X/Y)$$

  is the crossed product $\boxed{\mathscr{C}_X = \mathscr{C}_X(\mathcal{S}) := \mathcal{C}_X(\mathcal{S}) \rtimes X = \mathcal{C}_X(\mathcal{S}) \cdot C^*(X)}$
- $\mathscr{C}_X(\mathcal{S}) = \sum^{c}_{Y \in \mathcal{S}} \mathcal{C}_o(X/Y) \rtimes X \equiv \sum^{c}_{Y \in \mathcal{S}} \mathscr{C}_X(Y)$
- If we set $\mathscr{C}_X(Y) := \mathcal{C}_o(X/Y) \rtimes X = \mathcal{C}_o(X/Y) \cdot C^*(X)$ then we have
  - $\sum_{Y \in \mathcal{S}} \mathscr{C}_X(Y)$ is a direct linear sum
  - $\mathscr{C}_X(Y)\mathscr{C}_X(Z) \subset \mathscr{C}_X(Y \cap Z)$

Lattice of partitions of a 4-body system



C^*-algebras graded by semilattices

- Semilattice: ordered set $(\mathcal{S}, \leq)$ such that $\sigma \wedge \tau$ exists $\forall \sigma, \tau$
- $\mathscr{C} = \mathcal{S}$-graded C^*-algebra if a linearly independent family of C^*-subalgebras $\{\mathscr{C}(\sigma)\}_{\sigma \in \mathcal{S}}$ of $\mathscr{C}$ is given such that
  - $\sum_{\sigma \in \mathcal{S}}^{c} \mathscr{C}(\sigma) = \mathscr{C}$
  - $\mathscr{C}(\sigma)\mathscr{C}(\tau) \subset \mathscr{C}(\sigma \wedge \tau)\ \forall \sigma, \tau$
- $\mathscr{C}_{\geq \sigma} := \sum_{\tau \geq \sigma}^{c} \mathscr{C}(\tau)$ (graded) C^*-subalgebra
- $\mathscr{C}_{\not\geq \sigma} := \sum_{\tau \not\geq \sigma}^{c} \mathscr{C}(\tau)$ (graded) ideal
- $\mathscr{C} = \mathscr{C}_{\geq \sigma} + \mathscr{C}_{\not\geq \sigma}$ linear direct sum
- $\mathscr{P}_{\geq \sigma} : \mathscr{C} \to \mathscr{C}_{\geq \sigma}$ projection determined by this direct sum decomposition
- $\mathscr{P}_{\geq \sigma}$ is a morphism

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Abstract HVZ Theorem

$\mathcal{S}$ is atomic if $\exists\, o \equiv \min \mathcal{S}$ and each $\sigma \neq o$ is minorated by an atom; $\mathcal{P}(\mathcal{S}) =$ set of atoms of $\mathcal{S}$.

Theorem

If $\mathcal{S}$ is atomic then $\mathscr{P}T = (\mathscr{P}_{\geq\alpha}T)_{\alpha\in\mathcal{P}(\mathcal{S})}$ defines a morphism

$$\mathscr{P} : \mathscr{C} \to \prod\nolimits_{\alpha\in\mathcal{P}(\mathcal{S})} \mathscr{C}_{\geq\alpha} \quad \textit{such that } \ker \mathscr{P} = \mathscr{C}(o)$$

This gives us a canonical embedding

$$\mathscr{C}/\mathscr{C}(o) \subset \prod\nolimits_{\alpha\in\mathcal{P}(\mathcal{S})} \mathscr{C}_{\geq\alpha}$$

Corollary (Abstract HVZ Theorem)

Assume that $\mathscr{C} \subset L(\mathcal{H})$ and $\mathscr{C}(o) = K(\mathcal{H})$. If $H \in' \mathscr{C}$ and if we set $H_\alpha = \mathscr{P}_\alpha(H)$ then

$$\mathrm{Sp}_{\mathrm{ess}}(H) = \overline{\bigcup}_{\alpha\in\mathcal{P}(\mathcal{S})} \mathrm{Sp}(H_\alpha)$$

Many-body systems (M.Damak, VG: *Spectral analysis of many-body systems* JFA 259 (2010) 618–689)

- $\mathcal{S}$ = set of finite dimensional *compatible* real vector spaces equipped with translation invariant measures and such that: $X, Y \in \mathcal{S} \Longrightarrow X \cap Y \in \mathcal{S}$ (compatible $\Leftrightarrow$ subspaces of a real vector space)
- $X \in \mathcal{S}$ is the configuration spaces of an N-body system: $\mathcal{S}_X = \{Y \in \mathcal{S} \mid Y \subset X\}$
- If $O = \{0\} \in \mathcal{S}$ then $\mathcal{H}(O) = \mathbb{C}$ = vacuum state space
- The formalism will provide a mathematical framework for the description of the system obtained by coupling these subsystems
- The Hilbert space of the X system is: $\mathcal{H}(X) \equiv L^2(X)$
- Hilbert space of the total system: $\boxed{\mathcal{H} \equiv \mathcal{H}_{\mathcal{S}} = \oplus_X \mathcal{H}(X)}$

- $\mathscr{L}_{XY} = L(\mathcal{H}(Y), \mathcal{H}(X)), \quad \mathscr{K}_{XY} = K(\mathcal{H}(Y), \mathcal{H}(X))$
- $\mathscr{L}_X = \mathscr{L}_{XX}, \quad \mathscr{K}_X = \mathscr{K}_{XX}$
- $T \in L(\mathcal{H}) \Rightarrow T \cong (T_{XY})_{X,Y\in\mathcal{S}}$ with $T_{XY} \in \mathscr{L}_{XY}$

Man-body Hamiltonian algebra

- Let $X, Y \in \mathcal{S}$. If Z is a space with $X \cup Y \subset Z$ and $\varphi \in \mathcal{C}_c(Z)$ then
$$(T_{XY}(\varphi)u)(x) = \int_Y \varphi(x-y)u(y)\mathrm{d}y$$
defines a continuous operator $T_{XY}(\varphi) : \mathcal{H}(Y) \to \mathcal{H}(X)$.
- $\mathscr{T}_{XY} =$ closure in $\mathscr{L}_{XY}$ of the set of operators $T_{XY}(\varphi)$
This space is independent of the choice of Z and
$$\mathscr{T}_{XX} \equiv \mathcal{C}^*(X) = \text{group } C^*\text{-algebra of } X.$$
- $\mathscr{T} = (\mathscr{T}_{XY})_{X,Y\in\mathcal{S}} \subset L(\mathcal{H})$ closed self-adjoint subspace (closure of finite submatrices).
- **Theorem:** $\boxed{\mathscr{C} = \mathscr{T}^2 \text{ is a } C^*\text{-algebra}}$
- **Definition:** $\mathscr{C}$ is the **Hamiltonian algebra** of the system.
- **Theorem**
  - Assume that the ambient space is a real Hilbert space
  - Denote $X/Y = X \ominus (X \cap Y)$ and $\mathscr{K}_{X/Z,Y/Z} = K(\mathcal{H}(Y/Z), \mathcal{H}(X/Z))$
  - Then $\boxed{\mathscr{C}_{XY} = \mathscr{C}_{X\cap Y} \otimes \mathscr{K}_{X/Y,Y/X}}$

N-body $\to$ Many-body

- For each X we have an N-body algebra on $\mathcal{H}(X)$:

$$\mathscr{C}_X := \mathcal{C}_X \cdot \mathcal{C}^*(X) = \sum^{c}_{Y \subset X} \mathscr{C}_X(Y) \text{ with } \mathscr{C}_X(Y) = \mathcal{C}_{\mathrm{o}}(X/Y) \cdot C^*(X)$$

- These algebras are on the diagonal of $\mathscr{C}$ i.e. $\mathscr{C}_{XX} = \mathscr{C}_X$
- The non-diagonal spaces $\mathscr{C}_{XY} = \mathcal{C}_X \cdot \mathscr{T}_{XY}$ are Hilbert $(\mathscr{C}_X, \mathscr{C}_Y)$-bimodules
- If $\mathcal{S}$ has a largest element X : $\mathscr{C}$ = imprimitivity algebra of a graded full Hilbert $\mathscr{C}_X$-module
- The components of the quotient $\mathscr{C}_X/\mathscr{C}_X(O)$ live and are non-degenerate algebras on $\mathcal{H}(X)$;
  this is why you do not see a change of Hilbert space in the N-body case
- Not for the many-body system: $\mathscr{C}_{\geq X}$ lives (non-degenerate) on $\mathcal{H}_{\geq X} = \oplus_{Y \supset X} \mathcal{H}(Y) \subsetneq \mathcal{H}$

Elementary many-body Hamiltonians: Pauli-Fierz

- Here assume that $\mathcal{S}$ is finite and for each $Y \subset X$ choose a complement X/Y in X
- $X = X/Y \oplus Y \Rightarrow \mathcal{H}(X) = \mathcal{H}(X/Y) \otimes \mathcal{H}(Y)$
- $\theta \in \mathcal{H}(X/Y) \Rightarrow a^*(\theta) : \mathcal{H}(Y) \to \mathcal{H}(X)$ defined by $u \mapsto \theta \otimes u$
- $\Phi_{XY} \subset \mathscr{L}_{XY}$ the set of such operators
- $\Phi_{XY} := \Phi^*_{YX}$ if $Y \supset X$ and $\Phi_{XY} := 0$ if X, Y not comparable
- $\Phi_{XY} \cdot \mathcal{C}^*(Y) = \mathcal{C}^*(X) \cdot \Phi_{XY} = \mathscr{C}_{XY}$ if X, Y are comparable
- $\Phi = (\Phi_{XY})_{X,Y \in \mathcal{S}} \subset L(\mathcal{H})$ closed self-adjoint subspace
- Pauli-Fierz Hamiltonian: $H = K + \phi$ with $K \in \oplus_X h_X(P)$ pure kinetic energy and $\phi \in \Phi$ symmetric "field operator"

Theorem: The C^*-algebra generated by the Pauli-Fierz operators coincides with $\mathscr{C}$

Non-relativistic many-body Hamiltonians

- Tensor factorizations:
  - The ambient space is Hilbert, i.e. the X are equipped with compatible Euclidean structures
  - $X \in \mathcal{S} \Rightarrow \mathcal{S}/X = \{Y/X \mid Y \supset X\}$ is a semilattice of subspaces of the same Hilbert space
  - So $\mathscr{C}_{\mathcal{S}/X}$ may be constructed and is similar to $\mathscr{C}$ but it lives on the smaller Hilbert space $\mathcal{H}_{\geq X}$
  - $\mathcal{H}_{\geq X} = \mathcal{H}(X) \otimes \mathcal{H}_{\mathcal{S}/X}$ and $\mathscr{C}_{\geq X} = \mathcal{C}^*(X) \otimes \mathscr{C}_{\mathcal{S}/X}$
- Non-relativistic many-body Hamiltonian of type $\mathcal{S}$: ($\mathcal{S}$ finite)
  - self-adjoint bounded from below operator on $\mathcal{H}$ (strictly) affiliated to $\mathscr{C} = \mathscr{C}_{\mathcal{S}}$
  - $\forall X\, \exists H_{\mathcal{S}/X}$ on $\mathcal{H}_{\geq X}$ s.t. $H_{\geq X} \equiv \mathscr{P}_{\geq X}(H) = \Delta_X \otimes 1 + 1 \otimes H_{\mathcal{S}/X}$
  - $X = \max \mathcal{S} \Rightarrow H_{\mathcal{S}/X} = 0$ (Remark: $\mathcal{H}_{\mathcal{S}/\max \mathcal{S}} = \mathcal{H}(O) = \mathbb{C}$)
- Example: $K = \oplus_X \Delta_X$ and $I_{XY}(Z) = 1 \otimes I^Z_{XY}$

Non-relativistic many-body: Mourre estimate

- Conjugate operator:
  - $D = \oplus_X D_X, \quad D_X = (Q_X \cdot P_X + P_X \cdot Q_X)/4$
  - $U_\tau = \mathrm{e}^{\mathrm{i}\tau D}$ (the dilation group) $\Rightarrow U_\tau^* \mathscr{C} U_\tau = \mathscr{C}$
- H = non-relativistic $\Rightarrow \tau(H) := \bigcup_{X \neq O} \mathrm{ev}(H_{\mathcal{S}/X})$ threshold set
- $A \subset \mathbb{R} \Rightarrow N_A(\lambda) := \sup\{x \in A \mid x \leq \lambda\}$. Thus $N_A : \mathbb{R} \to [-\infty, +\infty[$

Theorem: $H =$ non-relativistic many-body Hamiltonian with $[D, (H+i)^{-1}] \in \mathscr{C}$. Then: $\tau(H)$ is a closed countable set and $\widehat{\rho}(\lambda) = \lambda - N_{\tau(H)}(\lambda)$. Hence $\widehat{\rho}(\lambda) > 0$ outside $\tau(H)$.

Sketch of proof: Write $X = \sigma$, $H_{\geq X} = H_\sigma$, $\rho_\sigma = \rho_{H_\sigma}$, $\mathcal{P} =$ set of atoms of $\mathcal{S}$

- $\widehat{\mathscr{C}} \subset \bigoplus_{\sigma \in \mathcal{P}} \mathscr{C}_\sigma \quad \widehat{H} = \bigoplus_{\sigma \in \mathcal{P}} H_\sigma \quad \widehat{\rho}_H = \inf_{\sigma \in \mathcal{P}} \rho_\sigma$
- $H_\sigma = \Delta_X \otimes 1 + 1 \otimes H_{\mathcal{S}/\sigma} \quad \rho_\sigma(\lambda) = \inf_\mu (\rho_{\Delta_X}(\mu) + \rho_{\mathcal{S}/\sigma}(\lambda - \mu))$
- $\rho_{\Delta_X}(\mu) = +\infty$ if $\mu < 0$ and $\rho_{\Delta_X}(\mu) = \mu$ if $\mu \geq 0$
- $\rho =$ l.s.c. Hence for $\sigma > o$: $\rho_\sigma(\lambda) = 0 \Leftrightarrow \rho_{\mathcal{S}/\sigma}(\lambda) \Leftrightarrow \lambda \in \tau_{H_{\mathcal{S}/\sigma}}$ (induction hypothesis)
- Thus $\widehat{\rho}_H(\lambda) = 0 \Leftrightarrow \exists \sigma \in \mathcal{P}$ such that $\lambda \in \tau_{H_{\mathcal{S}/\sigma}}$

Quantum fields (VG: *Spectral analysis of quantum field Hamiltonians* JFA 245 (2007) 89–143)

- $\mathcal{H}$ = complex Hilbert space (one particle space)
- $\Gamma(\mathcal{H})$ = symmetric or antisymmetric Fock space
- $\mathcal{O} \subset L(\mathcal{H})$ abelian C^*-algebra with $\overline{\mathcal{O}}^s \cap K(\mathcal{H}) = \{0\}$

  ($h = h^* \in' \mathcal{O}$ one-particle kinetic energy operators; $d\Gamma(h)$ kinetic energy operators of the field)
- The **algebra of QFH** determined by $\mathcal{O}$:

$$\boxed{\mathscr{F}(\mathcal{O}) = C^*(e^{i\phi(u)}\Gamma(A) \mid u \in \mathcal{H}, A \in \mathcal{O}, \|A\| < 1)}$$

Remark: $\mathscr{K}(\mathcal{H}) \equiv K(\Gamma(\mathcal{H})) \subset \mathscr{F}(\mathcal{O})$

Theorem: There is a unique morphism $\mathcal{P} : \mathscr{F}(\mathcal{O}) \to \mathcal{O} \otimes \mathscr{F}(\mathcal{O})$ such that for all $u \in \mathcal{H}$ and $A \in \mathcal{O}$ with $\|A\| < 1$

$$\mathcal{P}\left(e^{i\phi(u)}\Gamma(A)\right) = A \otimes \left(e^{i\phi(u)}\Gamma(A)\right).$$

We have $\ker \mathcal{P} = \mathscr{K}(\mathcal{H})$, so $\boxed{\widehat{\mathscr{F}}(\mathcal{O}) \equiv \mathscr{F}(\mathcal{O})/\mathscr{K}(\mathcal{H}) \hookrightarrow \mathcal{O} \otimes \mathscr{F}(\mathcal{O})}$

Quantum fields: Elementary and Standard Hamiltonians

- **Kinetic energy of the field:** $d\Gamma(h)$ if the one-particle kinetic energy is $h = h^* \in' \mathcal{O}$
- **Elementary QFI**: $V = V^* = \int_{\mathcal{H}} \mathrm{e}^{\mathrm{i}\phi(u)} d\mu(u)$, $\mu =$ bounded measure
- **Elementary QFH:** $H = d\Gamma(h) + V$ with $\boxed{h \geq m > 0,\ h^{-1} \in \mathcal{O};\ V = \text{elementary QFI}}$

Proposition: If $\mathcal{O}$ is non-degenerate on $\mathcal{H}$ then $\mathscr{F}(\mathcal{O})$ is the C^*-algebra generated by the elementary QFH and the "canonical morphism" $\mathcal{P}$ is characterized by:

$$H = d\Gamma(h) + V \text{ elementary } \Rightarrow \mathcal{P}(H) = h \otimes 1 + 1 \otimes H$$

Or, equivalently: $\mathcal{P}(\mathrm{e}^{-H}) = \mathrm{e}^{-h} \otimes \mathrm{e}^{-H}$.

- **Standard QF Hamiltonian:** $H = H^*$ on $\Gamma(\mathcal{H})$ is a **SQFH** if:
  - H is bounded from below and affiliated to $\mathscr{F}(\mathcal{O})$
  - $\exists\ h \in' \mathcal{O}$ self-adjoint on $\mathcal{H}$ with $\inf h = m > 0$ such that $\mathcal{P}(H) = h \otimes 1 + 1 \otimes H$

Quantum fields: essential spectrum of QFH

- **QFH of type** $\mathcal{O}$:= observable H affiliated to $\mathscr{F}(\mathcal{O}) \quad \Longrightarrow$
- $\widehat{H} := \mathcal{P}(H)$ observable affiliated to $\mathcal{O} \otimes \mathscr{F}(\mathcal{O}) \cong C_0(\mathrm{Sp}(\mathcal{O}); \mathscr{F}(\mathcal{O})) \quad \Longrightarrow$
- $\widehat{H} \cong \{\widehat{H}(k)\}_{k \in \mathrm{Sp}(\mathcal{O})}$ with $\widehat{H}(k)$ sadj ops on $\Gamma(\mathcal{H})$
- **Proposition:** Then $\mathrm{Sp}_{\mathrm{ess}}(H) = \mathrm{Sp}(\mathcal{P}(H)) = \bigcup_k \mathrm{Sp}(\widehat{H}(k))$
- **Proposition:** If H is a **SQFH** with one particle kinetic energy h:

  (i) $\mathrm{Sp}_{\mathrm{ess}}(H) = \mathrm{Sp}(h) + \mathrm{Sp}(H)$ $\quad \Big($Hence $\mathrm{Sp}_{\mathrm{ess}}(H) = [m + \inf H, \infty[$ if $\mathrm{Sp}(h) = [m, \infty[\,\Big)$

  (ii) $\inf H$ is an eigenvalue of H of finite multiplicity and isolated from the rest of the spectrum.

Quantum fields: Mourre estimate, framework

One particle conjugate operator: $\mathfrak{a} = \mathfrak{a}^*$ on $\mathcal{H}$ such that:

- $\mathrm{e}^{-\mathrm{i}\tau\mathfrak{a}}\mathcal{O}\mathrm{e}^{\mathrm{i}\tau\mathfrak{a}} = \mathcal{O} \ \ \forall\tau$
- $\tau \mapsto \mathrm{e}^{-\mathrm{i}\tau\mathfrak{a}}S\mathrm{e}^{\mathrm{i}\tau\mathfrak{a}}$ is norm continuous $\forall S \in \mathcal{O}$

Field conjugate operator: $A = \mathrm{d}\Gamma(\mathfrak{a})$ so $\mathrm{e}^{\mathrm{i}\tau A} = \Gamma(\mathrm{e}^{\mathrm{i}\tau\mathfrak{a}})$. Then:

- $\mathrm{e}^{-\mathrm{i}\tau A}\mathscr{F}(\mathcal{O})\mathrm{e}^{\mathrm{i}\tau A} = \mathscr{F}(\mathcal{O})$
- $\tau \mapsto \mathrm{e}^{-\mathrm{i}\tau A}T\mathrm{e}^{\mathrm{i}\tau A}$ is norm continuous for all $T \in \mathscr{F}(\mathcal{O})$.
- if $\widehat{A} \equiv \mathfrak{a} \otimes 1 + 1 \otimes A$ on $\mathcal{H} \otimes \Gamma(\mathcal{H})$ then for all $T \in \mathscr{F}(\mathcal{O})$
$$\mathcal{P}(\mathrm{e}^{-\mathrm{i}\tau A}T\mathrm{e}^{\mathrm{i}\tau A}) = \mathrm{e}^{-\mathrm{i}\tau\widehat{A}}\mathcal{P}(T)\mathrm{e}^{\mathrm{i}\tau\widehat{A}}$$

- Thus in the general formalism we have:
$$\mathscr{C} = \mathscr{F}(\mathcal{O}), \quad \mathcal{U}_\tau \text{ is implemented by } \mathrm{e}^{\mathrm{i}\tau A}, \quad \widehat{\mathcal{U}}_\tau \text{ is implemented by } \mathrm{e}^{\mathrm{i}\tau\widehat{A}}$$

Quantum fields: Mourre estimate, sketch of proof

- $H \in' \mathscr{F}(\mathcal{O})$, $H \geq 0$, $\widehat{H} = h \otimes 1 + 1 \otimes H$, $h \geq m > 0$
- $\widehat{\rho}_H = \rho_{\widehat{H}}$ and $\rho_h(x) = \infty$ if $x < m$ and $\rho_H(x) = \infty$ if $x < 0$

$$\boxed{\widehat{\rho}_H(\lambda) = \inf_{0 \leq \mu \leq \lambda - m} \Big(\rho_h(\lambda - \mu) + \rho_H(\mu) \Big)}$$

- Corollary: If $\rho_h \geq 0$ then $\rho_H \geq 0$ hence $\tau(H) = \varkappa(h) + \varkappa(H)$
- Recall: $\varkappa(H) = \tau(H) \cup \sigma_{\mathrm{p}}(H)$. Clearly: $\varkappa(h) = \tau(h)$.
- **Conclusion: we get an equation for the unknown set $\tau(H)$:**

$$\tau(H) = [\tau(h) + \sigma_{\mathrm{p}}(H)] \bigcup [\tau(h) + \tau(H)]$$

- Solve by iteration and by using $X + Y \cup Z = [X + Y] \cup [X + Z]$

Quantum fields: Mourre estimate for Standard QFH

Theorem: $H =$ SQFH of type $\mathcal{O}$ with one particle kinetic energy h such that

- h is of class $C^1_{\mathrm{u}}(\mathfrak{a})$ and $[h, \mathrm{i}\mathfrak{a}] \geq 0$
- H is of class $C^1_{\mathrm{u}}(A)$

Denote $\tau^n(h) = \tau(h) + \cdots + \tau(h)$. Then: $\boxed{\tau(H) = \left[\bigcup_{n=1}^{\infty} \tau^n(h)\right] + \sigma_{\mathrm{p}}(H)}$

Remarks:

- $H_0 = \mathrm{d}\Gamma(h) \Rightarrow \tau(H_0) = \bigcup_{n=1}^{\infty} \tau^n(h)$
- $\boxed{\tau(H) = \tau(H_0) + \sigma_{\mathrm{p}}(H)}$

Graded Weyl algebra (ABM+VG 1993 and VG+AI 2001)

- This covers N-body systems (with magnetic fields) and quantum fields.
- Introduced in

  ABM+VG *"Graded C^*-algebras associated to symplectic spaces and spectral analysis of many channel Hamiltonians"* (1993)

  VG+AI *"C^*-algebras of energy observables: II. Graded symplectic algebras and magnetic Hamiltonians"* (2001)
- Independently considered (for different reasons) in

  Buchholz+Grundlingen *"The resolvent algebra: new approach to canonical quantum systems"* (JFA 2008)
- (Ξ, σ) finite dimensional symplectic space (the passage to infinite dimensions is trivial for this algebra). Let $(W, \mathcal{H})$ be a representation of the CCR: $W(\xi)W(\eta) = \mathrm{e}^{\mathrm{i}\sigma(\xi,\eta)/2}W(\xi+\eta)$
- For each integrable Borel measure on Ξ set $W(\mu) = \int_\Xi W(\xi)\mu(\mathrm{d}\xi)$
- For each subspace $E \subset \Xi$ let $\mathscr{C}(E)$ be the norm closure of the space of $W(\mu)$ such that $\operatorname{supp}\mu \subset E$ and μ is E-absolutely continuous. Then
  - $\mathscr{C}(E)$ is a C^*-algebra
  - $\mathscr{C}(E)\mathscr{C}(F) \subset \mathscr{C}(E+F)$
  - $\{\mathscr{C}(E)\}_{E\subset\Xi}$ is linearly independent
- Thus the graded Weyl algebra $\mathscr{C}_\Xi = \sum^{\mathrm{c}}_{E\subset\Xi}\mathscr{C}(E)$ is a C^*-algebra graded by the set of all subspaces of Ξ equipped with the order relation $E \le F \Leftrightarrow E \supset F$.

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- VG: *On the structure of the essential spectrum of elliptic operators on metric spaces* (2010)

Complements

Examples of operators affiliated to the Hamiltonian algebras:

(1) N-body systems

(2) Many-body systems

(3) Quantum fields

Affiliattion criteria

- $\mathscr{C} \subset L(\mathcal{H})$ C^*-subalgebra
- A self-adjoint operator H on $\mathcal{H}$ is affiliated to $\mathscr{C}$ if it satisfies

$$(H-z)^{-1} \in \mathscr{C} \text{ for some } z \quad \left(\iff \theta(H) \in \mathscr{C} \quad \forall \theta \in C_0(\mathbb{R})\right)$$

- **Affiliation criteria:** Let $H_0 \geq 1$ self-adjoint affiliated to $\mathscr{C}$
  - V quadratic form with $-aH_0 - b \leq V \leq bH_0$ for some $a < 1$ and $b > 0$
    - $H = H_0 + V$ (form sum) is a self-adjoint operator and
    - $\exists \alpha > 1/2$ such that $H_0^{-\alpha} V H_0^{-1/2} \in \mathscr{C} \Longrightarrow H$ affiliated to $\mathscr{C}$.
  - V self-adjoint bounded from below such that $H = H_0 + V$ is self-adjoint on $D(H_0) \cap D(V)$

$$\mathrm{e}^{-tH_0}\mathrm{e}^{-2tV}\mathrm{e}^{-tH_0} \in \mathscr{C} \ \forall t > 0 \Longrightarrow H \text{ is affiliated to } \mathscr{C}.$$

- A norm resolvent limit of self-adjoint operators affiliated to $\mathscr{C}$ is affiliated to $\mathscr{C}$.

N-body system - physical setting

- N "elementary particles" with masses $m_1, \dots, m_N$
- Configuration space (center of mass ref. system, no external forces)
  $X = \{x = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N \mid \sum_k m_k x_k = 0\}$
- $\langle x|y\rangle = \sum_{k=1}^N 2m_k x_k y_k$ such that $\Delta_X \sim \sum_k \frac{1}{2m_k} P_k^2$
- Hamiltonian: $H = \Delta_X + \sum_{i<k} V_{ik}(x_i - x_k)$ on $L^2(X)$
- Cluster decomposition $\equiv$ partition σ of the set $\{1, \dots, N\}$
  - $|\sigma| =$ number of elements of σ
  - Cluster $\equiv$ element of the partition σ
    $\equiv$ "composite particle" of mass $m_a = \sum_{k \in a} m_k$.
- Think of σ as a system of $|\sigma|$ particles with masses m_a

N-body subsystems

- Configuration space of the σ system:

$$\begin{aligned} X_\sigma &= \{x = (x_a)_{a\in\sigma} \in (\mathbb{R}^d)^{|\sigma|} \mid \textstyle\sum_a m_a x_a = 0\} \\ &\cong \{x \in X \mid x_i = x_j \text{ if } i, j \text{ belong to the same cluster of } \sigma\} \end{aligned}$$

  gives an embedding $X_\sigma \hookrightarrow X$

- Equip X_σ with the scalar product induced by X:

  $\langle x|y\rangle = \sum_{a\in\sigma} 2m_a x_a y_a$ then $\Delta_{X_\sigma} \sim \sum_a \frac{1}{2m_a} P_a^2$

- The set $\mathfrak{S}$ of partitions is ordered and lower bounds exist:
  - $\sigma \leq \tau \iff \tau$ is finer than σ
  - $\sigma \leq \tau \iff X_\sigma \subset X_\tau$
  - $X_\sigma \cap X_\tau = X_{\sigma\wedge\tau}$
  - $\mathfrak{S} \cong \mathcal{S} = \{X_\sigma \mid \sigma \in \mathfrak{S}\}$ semilattice of subspaces of X

The N-body C^*-algebra: physical translation invariant case

- $H = \Delta_X + \sum_{i<k} V_{ik}(x_i - x_k)$
- $X \ni x \mapsto x_i - x_k \in \mathbb{R}^d$ is surjective and has $X_{(ik)}$ as kernel
- $H = \Delta_X + \sum_{\sigma \in \mathfrak{S}_2} V_\sigma \circ \pi_\sigma$ with $V_\sigma : X/X_\sigma \to \mathbb{R}$
- $\mathcal{C}_X(X_\sigma) := \mathcal{C}_{\mathrm{o}}(X/X_\sigma) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \Rightarrow H = \Delta_X + \sum_{\sigma \in \mathfrak{S}_2} V_\sigma$
- $\mathcal{C}_X := \sum_\sigma \mathcal{C}_X(X_\sigma)$ sum over all partitions σ
- $\mathcal{C}^*(X) :=$ the group C^*-algebra (C_0 functions of $P_X = -i\nabla_X$)

Theorem

The C^*-algebra generated by the operators $\mathcal{V}_k^* H \mathcal{V}_k$, i.e. by

$$(P_X + k)^2 + \sum_{\sigma \in \mathfrak{S}_2} V_\sigma \quad \textit{with } V_\sigma \in \mathcal{C}_X(X_\sigma), k \in X^*$$

$$\textit{is} \quad \mathscr{C}_X = \mathcal{C}_X \cdot \mathcal{C}^*(X) = \sum_\sigma \mathcal{C}_X(X_\sigma) \cdot \mathcal{C}^*(X) =: \sum_\sigma \mathscr{C}_X(X_\sigma)$$

N-body Hamiltonians: examples

- $h : X \to \mathbb{R}$ continuous with $c'|x|^{2s} \leq h(x) \leq c''|x|^{2s}$ (some $s > 0$ and $|x|$ large)
- $H(\max \mathfrak{S}) = h(P)$ self-adjoint on $\mathcal{H}$ and $D(|H(X)|^{1/2}) = \mathcal{H}^s$ (Sobolev space)
- If $\sigma \neq \max \mathfrak{S}$ let $H(\sigma) : \mathcal{H}^s \to \mathcal{H}^{-s}$ symmetric, $H(\sigma) \geq -\mu_\sigma h(P) - \nu$, $\mu_\sigma \geq 0$, $\sum \mu_\sigma < 1$
- Then $H = \sum_\tau H(\tau)$ and $H_{\geq \sigma} = \sum_{\tau \geq \sigma} H(\tau)$ are self-adjoint operators on $\mathcal{H}$.
- Let $t > s$ and $\| \cdot \|_{s,t}$ the norm in $L(\mathcal{H}^s, \mathcal{H}^{-t})$. Assume:

  (1) $U_x^* H(\sigma) U_x = H(\sigma)$ if $x \in X_\sigma$; $\quad \big(U_x = \text{translation by } x \big)$

  (2) $\|V_k^* H(\sigma) V_k - H(\sigma)\|_{s,t} \to 0$ if $k \to 0$ in X_σ; $\quad \big(V_k(x) = \mathrm{e}^{\mathrm{i}\langle k, x \rangle} \big)$

  (3) $\|(V_k - 1) H(\sigma)\|_{s,t} \to 0$ if $k \to 0$ in $X_\sigma^\perp$.
- Then H is affiliated to $\mathscr{C}_X$ and $H_{\geq \sigma} = \mathcal{P}_{\geq \sigma}(H)$

Unfolding an N-body system

Want to allow an N-body system to make transitions to one of its subsystems in finite time and to allow transitions between different subsystem by creation-annihilation processes as in field theory.

- Hilbert state space of the system $\mathcal{H} := \oplus_\sigma \mathcal{H}(X_\sigma)$.
- State $u \in \mathcal{H}(X_\sigma) \Rightarrow$ we have $|\sigma|$ particles of masses m_a.
- $\sigma = \min \mathfrak{S} = \{\{1, \dots, N\}\} \Rightarrow$ get one particle with mass $M = m_1 + \cdots + m_N$.
  There are no external fields $\Rightarrow$ we are in the vacuum state: $\mathcal{H}(X_{\min \mathfrak{S}}) = \mathbb{C}$.
- Would like to treat usual inter-cluster interactions associated e.g. to potentials defined on $X^\sigma = X/X_\sigma$ but also interactions which force the system to make a transition from a "phase" σ to a "phase" τ
  - a system of $|\sigma|$ particles with masses $(m_a)_{a\in\sigma}$ is tranformed into a system of $|\tau|$ particles with masses $(m_b)_{b\in\tau}$
  - thus the number of particles varies from 1 to N but the total mass existing in the "universe" is constant and equal to M.

Many-body Hamiltonian: formal structure

- $H = (H_{XY})_{X,Y\in\mathcal{S}}$ where $H_{XY} : D(H_{XY}) \subset \mathcal{H}(Y) \to \mathcal{H}(X)$ and $H^*_{XY} = H_{YX}$
- $H = K + I \Rightarrow H_{XY} = K_{XY} + I_{XY}$ (K = kinetic energy and I = interaction)
- K diagonal: $K_{XY} = 0$ if $X \neq Y$ and $K_{XX} \equiv K_X$ (kinetic energy of system X)
- $K_X = h_X(P)$ with $h_X : X^* \to \mathbb{R}$ continuous and $|h_X(k)| \to \infty$ if $k \to \infty$
- If $\mathcal{S}$ is infinite we need $\lim_X \inf_k |h_X(k)| = \infty$ (non-zero mass in QFT)
- $I_{XY} = \sum_{Z\subset X\cap Y} I_{XY}(Z)$ will reflect the N-body structures of X, Y
- $H_{XX} = K_X + I_{XX}$ Hamiltonian of system X, N-body type (N depends on X)
- $H_{XY} = I_{XY}$ for $X \neq Y$ is the interaction between the systems X and Y
- Simplest interaction: $I \in M(\mathscr{C})$. Then $H = K + I$ is strictly affiliated to $\mathscr{C}$ and $\mathscr{P}_{\geq X}(H) = K_{\geq X} + \mathscr{P}_{\geq X}(I)$ where $K_{\geq X} = \oplus_{Y\geq X} K_Y$

Many-body Hamiltonian: example

- $\mathcal{H}^s(X)$ usual Sobolev spaces
- Assume: $D(|K_X|^{1/2}) = \mathcal{H}^s(X)$ and $D(|K_Y|^{1/2}) = \mathcal{H}^t(Y)$
- Define $I_{XY}(Z)$ by the relation

$$\mathcal{F}_Z I_{XY}(Z)\mathcal{F}_Z^{-1} \equiv \int_Z^{\oplus} I_{XY}^Z(k)\mathrm{d}k$$

where $I_{XY}^Z : Z \to L(\mathcal{H}^t(Y/Z), \mathcal{H}^{-s}(X/Z))$ is continuous and

$$\sup_k \|(1+|k|+|P_{X/Z}|)^{-s} I_{XY}^Z(k)(1+|k|+|P_{Y/Z}|)^{-t}\| < \infty.$$

- The operators $I_{XY}^Z(k)$ must decay in a weak sense at infinity: $\exists \varepsilon > 0$

$$I_{XY}^Z(k) : \mathcal{H}^t(Y/Z) \to \mathcal{H}^{-s-\varepsilon}(X/Z) \text{ is compact}$$

- Particular case: $I_{XY}(Z) = 1_Z \otimes I_{XY}^Z$ with $I_{XY}^Z : \mathcal{H}^t(Y/Z) \to \mathcal{H}^{-s}(X/Z)$ compact

Many-body Hamiltonian: more on the definition of H

- $K = \oplus_X K_X$ total kinetic energy $\Rightarrow \mathcal{G} = D(|K|^{1/2})$ its form domain
- The matrix $I(Z) = (I_{XY}(Z))_{X,Y\in\mathcal{S}}$ can be realized as a linear operator $\mathcal{G}_\mathrm{o} \to \mathcal{G}_\mathrm{o}^*$ where $\mathcal{G}_\mathrm{o}$ is the algebraic direct sum of the spaces $\mathcal{G}(X)$. We require that this extend to a continuous map $I(Z) : \mathcal{G} \to \mathcal{G}^*$
- $I(Z)$ is norm limit in $L(\mathcal{G}, \mathcal{G}^*)$ of its finite sub-matrices $(I_{XY}(Z))_{X,Y\in\mathcal{T}}$
- There are $\mu_Z \geq 0$ and $a \geq 0$ with $\sum_Z \mu_Z < 1$ such that $I(Z) \geq -\mu_Z|K + ia|\ \forall Z$
- The series $\sum_Z I(Z) \equiv I$ is norm summable in $L(\mathcal{G}, \mathcal{G}^*)$.

Then the Hamiltonian defined as a form sum $H = K + I$ is a self-adjoint operator strictly affiliated to $\mathscr{C}$, we have $H_{\geq X} = K_{\geq X} + \sum_{Z\geq X} I(Z)$, and the essential spectrum of H is given by

$$\mathrm{Sp_{ess}}(H) = \overline{\bigcup}_{X\in\mathcal{P}(\mathcal{S})} \mathrm{Sp}(H_{\geq X})$$

Remark: The set of self-adjoint operators affiliated to $\mathscr{C}$ is stable under norm resolvent limits. The formula is valid for all such operators ($H_{\geq X}$ less explicit).

Many-body: short range many-body forces

- $H = \oplus_X \Delta_X + (I_{XY})_{X,Y\in\mathcal{S}}$ and $I_{XY} = \sum_{Z\subset X\cap Y} I_{XY}(Z)$
- $Z \subset X \cap Y \Rightarrow X = Z \oplus (X/Z)$ and $Y = Z \oplus (Y/Z)$ hence

$$\mathcal{H}(X) = \mathcal{H}(Z) \otimes \mathcal{H}(X/Z) \quad \text{and} \quad \mathcal{H}(Y) = \mathcal{H}(Z) \otimes \mathcal{H}(Y/Z)$$

- $I_{XY}(Z) = 1_Z \otimes I^Z_{XY}$ with $I^Z_{XY} : \mathcal{H}^2(Y/Z) \to \mathcal{H}(X/Z)$
- $E = (X \cap Y)/Z \Longrightarrow Y/Z = E \oplus (Y/X)$ and $X/Z = E \oplus (X/Y)$
- $(I^Z_{XY} u)(x') = \int_{Y/X} I^Z_{XY}(x', y') u(y') \mathrm{d}y'$
- $X \boxplus Y = X/Y \times Y/X \Rightarrow I^Z_{XY} : X \boxplus Y \to K(\mathcal{H}^2(E), \mathcal{H}(E)) \equiv \mathscr{K}^2(E)$
- Suffices to have $I^Z_{XY} \in L^2(X \boxplus Y; \mathscr{K}^2(E))$ for the affiliation to $\mathscr{C}$
- Short-range assumption which implies everything:

$$(\langle x' \rangle^r + \langle y' \rangle^r) \| I^Z_{XY}(x', y') \|_{\mathscr{K}^2(E)} + \| \langle Q_E \rangle^r I^Z_{XY}(x', y') \|_{\mathscr{K}^2(E)}$$

is integrable for some $r > 1$.

Quantum fields: $P(\varphi)_2$ models

- Here we consider only the bosonic case.
- $\mathcal{E} \subset \mathcal{H}$ is **isotropic** if it is a *real* linear subspace and $\Im(u, v) = 0$ for all $u, v \in \mathcal{E}$. A maximal isotropic subspace is called **Lagrangian**. Fix such a space.
- $\mathscr{W}(\mathcal{E})$ = Von Neumann algebra generated by the $\phi(u)$ with $u \in \mathcal{E}$
- $\mathscr{W}(\mathcal{E})$ is maximal abelian and Ω is a cyclic and separating vector. So $\langle T \rangle = \langle \Omega | T\Omega \rangle$ defines a faithful state on $\mathscr{W}$
- $L^p(\mathcal{E})$ associated to the couple $\mathscr{W}, \langle\cdot\rangle$. These are spaces of unbounded operators on $\Gamma(\mathcal{H})$ such that $\mathscr{W}(\mathcal{E}) = L^\infty(\mathcal{E}) \subset L^2(\mathcal{E}) = \Gamma(\mathcal{H}) \subset L^1(\mathcal{E})$

Theorem: Assume

- $H_0 = \mathrm{d}\Gamma(h)$ where $h \in' \mathcal{O}$ with $\inf h > 0$ and $\boxed{h^{-1}\mathcal{E} \subset \mathcal{E}}$
- $V \in \bigcap_{p<\infty} L^p(\mathcal{E})$ self-adjoint (hence affiliated to $\mathscr{W}(\mathcal{E})$)
- There is a sequence of operators $V_n \in \bigcap_{p<\infty} L^p(\mathcal{E})$ which are *bounded from below*, and there is $q > 2$ such that:

$$\sup_n \|\mathrm{e}^{-V_n}\|_{L^q} < \infty, \ \|V_n - V\|_{L^q} \to 0.$$

Then $H_0 + V$ is essentially self-adjoint on $D(H_0) \cap D(V)$ and its closure H is a SQFH of type $\mathcal{O}$ with one particle kinetic energy h.

Coupled systems

- $\mathscr{C}_1, \mathscr{C}_2$ nuclear on $\mathcal{H}_1, \mathcal{H}_2$ and $K(\mathcal{H}_k) \subset \mathscr{C}_k$
- $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \Longrightarrow \mathscr{C} = \mathscr{C}_1 \otimes \mathscr{C}_2 \subset L(\mathcal{H})$ and $K(\mathcal{H}) = K(\mathcal{H}_1) \otimes K(\mathcal{H}_2)$
- $\mathcal{P}_k : \mathscr{C}_k \to \widehat{\mathscr{C}_k} \equiv \mathscr{C}_k / K(\mathcal{H}_k)$ the canonical surjection
- $\mathcal{P}_1' = \mathcal{P}_1 \otimes \mathrm{id}_{\mathscr{C}_2} : \mathscr{C} \to \widehat{\mathscr{C}_1} \otimes \mathscr{C}_2 \qquad \mathcal{P}_2' = \mathrm{id}_{\mathscr{C}_1} \otimes \mathcal{P}_2 : \mathscr{C} \to \mathscr{C}_1 \otimes \widehat{\mathscr{C}_2}$
- Then the kernel of the next morphism is equal to $K(\mathcal{H})$:

  $$\mathcal{P} := \mathcal{P}_1' \oplus \mathcal{P}_2' : \mathscr{C} \to \left(\widehat{\mathscr{C}_1} \otimes \mathscr{C}_2\right) \oplus \left(\mathscr{C}_1 \otimes \widehat{\mathscr{C}_2}\right)$$
- $H \in' \mathscr{C} \Longrightarrow \widehat{H}_1 = \mathcal{P}_1'(H) \in' \widehat{\mathscr{C}_1} \otimes \mathscr{C}_2$ and $\widehat{H}_2 = \mathcal{P}_2'(H) \in' \mathscr{C}_1 \otimes \widehat{\mathscr{C}_2}$
- $\boxed{\mathrm{Sp}_{\mathrm{ess}}(H) = \mathrm{Sp}(\widehat{H}_1) \cup \mathrm{Sp}(\widehat{H}_2)}$
- **Example:** Field coupled with a “confined system” (e.g. Pauli-Fierz model):

  $n = 2$, $\mathscr{C}_2 = K(\mathcal{H}_2)$, $\mathcal{P}_2' = 0 \Longrightarrow \quad \mathrm{Sp}_{\mathrm{ess}}(H) = \mathrm{Sp}(\widehat{H}_1)$

Quantum fields: Positive mass Pauli-Fierz model

- $\mathscr{L}$ = Hilbert space of the confined system, $K(\mathscr{L})$ = its Hamiltonian algebra
- $\mathcal{H} = \Gamma(\mathcal{H}) \otimes \mathscr{L}$ = Hilbert space of the total system
- $\mathscr{F}(\mathcal{O}, \mathscr{L}) = \mathscr{F}(\mathcal{O}) \otimes K(\mathscr{L})$ = Hamiltonian algebra of the coupled system:

$$\mathscr{F}(\mathcal{O}, \mathscr{L})/K(\mathcal{H}) \hookrightarrow \mathcal{O} \otimes \mathscr{F}(\mathcal{O}, \mathscr{L})$$

- $H = \mathrm{d}\Gamma(h) \otimes 1 + 1 \otimes L + \phi(v) \equiv H_0 + \phi(v)$ total Hamiltonian such that:

  (1) $h \in' \mathcal{O}$ with $\inf h > 0$ (2) L positive with purely discrete spectrum

  (3) $v : D(L^{1/2}) \to D(h^{1/2})^* \otimes \mathscr{L}$ with $\lim_{r\to\infty} \|(h^{-1/2} \otimes 1)v(L+r)^{-1/2}\| < 1$.

  (4) $(h+L)^{-\alpha} v(L+1)^{-1/2}$ and $(h+L)^{-1/2} v(L+1)^{-\alpha}$ are compact if $\alpha > 1/2$.

(1) $H \in' \mathscr{F}(\mathcal{O}, \mathscr{L})$ and is a standard QFH with h as one particle kinetic energy
(2) $\sigma_{\mathrm{ess}}(H) = \sigma(h) + \sigma(H)$
(3) $A = \mathrm{d}\Gamma(\mathfrak{a}) \otimes 1$; H of class $C^1_{\mathrm{u}}(A)$ and h of class $C^1_{\mathrm{u}}(\mathfrak{a})$ with $[h, \mathrm{i}\mathfrak{a}] \geq 0 \Longrightarrow$

$$\tau(H) = \left[\bigcup_{n=1}^{\infty} \tau^n(h)\right] + \sigma_{\mathrm{p}}(H)$$