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① # The Brockett–Wegner Diagonalizing Flow

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I. Introduction

Let $H = H^*$, $A = A^* \in \mathrm{Mat}(N, \mathbb{C})$ be two s.a. complex $N \times N$-matrices.

H is called <u>A-diagonal</u> $:\Leftrightarrow [H, A] = 0$.

Study IVP
$$\dot{H}_t := i[H_t, G_t], \quad H_0 := H,$$
with $G_t \in C(\mathbb{R}_0^+, \mathrm{Mat}(N,\mathbb{C}))$ chosen later.

If U_t is the unitary solving
$$\dot{U}_t = i G_t U_t, \quad U_0 = \mathbb{1}$$
then
$$H_t = U_t^* H U_t \quad (\in C^1(\mathbb{R}_0^+, \mathrm{Mat}(N,\mathbb{C})))$$
is unitarily equivalent to H, $\forall t \geq 0$.

Define the Lyapunov function
$$f_t := \frac{1}{2} \mathrm{Tr}\{(H_t - A)^2\} \geq 0.$$
Note that
$$f_t = \frac{1}{2} \mathrm{Tr}\{H_t^2 + A^2 - 2 H_t A\}$$
$$= \frac{1}{2} \mathrm{Tr}\{H^2 + A^2\} - \mathrm{Tr}\{H_t A\}$$
and hence, by cyclicity of $\mathrm{Tr}(\cdot)$
$$\dot{f}_t = -\mathrm{Tr}\{\dot{H}_t A\} = -\mathrm{Tr}\{i[H_t, G_t] A\}$$
$$= -\mathrm{Tr}\{i[A, H_t] G_t\}.$$

So, choosing [Brockett '91]

$$G_t := i[A, H_t],$$

we obtain

$$\forall t \geq 0: \quad f_t \geq 0, \quad \dot{f}_t = -\mathrm{Tr}\{G_t^2\} \leq 0.$$

$$\Rightarrow \quad \mathrm{Tr}\{G_t^2\} = -\dot{f}_t \in L^1(\mathbb{R}_0^+).$$

$$\text{„}\Rightarrow\text{"} \quad \| [H_t, A] \|_{HS} \longrightarrow 0, \quad t \to \infty$$

- [Wegner '94]: similar idea with

$$G_t := i[H_t^{diag}, H_t]$$

flaws: • refers explicitly to a basis
• cubicly nonlinear in H

- [Deift, Li, Tomei '85]: Toda flow $\dot{L}_t = [B_t, L_t]$ with

$$L_t = \begin{pmatrix} a_1 & b_1 & & 0 \\ b_1 & a_2 & b_2 \ldots & \\ & b_2 & \ddots & b_{N-1} \\ 0 & & b_{N-1} & a_N \end{pmatrix}, \quad B_t = \begin{pmatrix} 0 & b_1 & & 0 \\ -b_1 & 0 & \ddots & \\ & \ddots & & b_{N-1} \\ 0 & & -b_{N-1} & 0 \end{pmatrix}$$

is B-W flow with $G_t = i[A, H_t]$, $H_t := L_t$, $A := \mathrm{diag}(1, 2, \ldots, N)$. $\longrightarrow$ integrable, related to QR-factorization

- [Kehrein (et al) 2006]: continuous unitary transformations („CUT") applied to various QFT models: Spin-Boson, Hubbard (with $A = \mathbb{D}$); Application is only formal

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Brockett's idea bears mathematical problems:

- $\dot{H}_t = [H_t, [H_t, A]]$, $H_0 = H$
  is nonlinear. Global Existence?
- f_t employs Tr on $Mat(N, \mathbb{C})$,
  $\Rightarrow$ For $\dim \mathcal{H} = \infty$: $f_0 < \infty \Leftrightarrow (H-A) \in \mathcal{L}^2(\mathcal{H})$
- $[H_t, A] \to 0$ does not imply
  - convergence of $H_t \to H_\infty$ nor
  - convergence of $U_t \to U_\infty$.

Thm 1 Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a C^*-subalg. of $\mathcal{B}(\mathcal{H})$ with unitarily invar. norm $\|\cdot\|_{\mathcal{A}}$. Suppose that $H = H^*$, $A = A^* \in \mathcal{A}$. Then the IVP

$$\forall t > 0: \quad \dot{H}_t = [H_t, [H_t, A]] , \quad H_0 = H \qquad (1)$$

has a unique sol'n $H_{(\cdot)} \in C^\infty(\mathbb{R}_0^+; \mathcal{A})$, and for all $t > 0$, $H_t \cong H$.

For the next statement, we assume that $A = A^* \in \mathcal{B}(\mathcal{D}, \mathcal{H})$ is a positive op. on $\mathcal{H}$ with $A \geq \mathbb{1}$ and denote $X := \mathcal{B}(\mathcal{H})$, $Y := \mathcal{B}(\mathcal{D})$, with $\|T\|_{\mathcal{D}} := \|A\,T\|_{\mathcal{H}}$.
We define recursively

$$R_0(H) := A^{-1}, \quad R_1(H) := [H, A^{-1}], \ldots$$
$$\ldots \quad R_n := [H, \cdots [H, A^{-1}] \cdots] = ad_H^n(A^{-1}).$$

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<u>Thm 2</u>: Suppose that $H = H^*$, $A = A^* \geq 1$ are two selfadjoint op's on $\mathcal{D} \subseteq \mathcal{H}$ and that $R_n(H) \in \mathcal{B}(\mathcal{H}) \cap \mathcal{B}(\mathcal{D})$, $\| HA^{-1} \|_{\mathcal{B}(\mathcal{H})} < \infty$ and

$$\sum_{n=0}^{\infty} \frac{e^{\varrho n}}{n!} \Big(\| R_n(H) \|_{\mathcal{B}(\mathcal{H})} + \| R_n(H) \|_{\mathcal{B}(\mathcal{D})} \Big) \leq e^{\eta}.$$

Then (1) has a unique s.a. solution of (1) $H_{(\cdot)} \in C^{\infty}([0, T_*) ; \mathcal{B}(\mathcal{D}, \mathcal{H}))$, with $T_* := \frac{1}{8} e^{\varrho - \eta - 1}$ and unitary family

$U_{(\cdot)} \in C^{\infty}([0, T_*) ; \mathcal{B}(\mathcal{H}) \cap \mathcal{B}(\mathcal{D}))$ s.th.

$H_t = U_t H U_t^*$. Moreover,

$$\sum_{n=0}^{\infty} \frac{e^{\varrho n}}{n!} \cdot \sup_{0 \leq t < T_*} \Big\{ \| R_n(H_t) \|_{\mathcal{B}(\mathcal{H})} + \| R_n(H_t) \|_{\mathcal{B}(\mathcal{D})} \Big\} \leq 2 e^{\eta}.$$

<u>Thm 3</u>: Suppose $H = H^*$, $A = A^* \in \mathcal{L}^2(\mathcal{H})$ such that $A > 0$ has full rank ($f \neq 0 \Rightarrow \langle f | A f \rangle > 0$). Let $H_{(\cdot)} \in C^{\infty}(\mathbb{R}_0^+ ; \mathcal{L}^2(\mathcal{H}))$ be the unique solution of (1). Then

$$H_t \longrightarrow H_\infty, \quad t \to \infty$$

strongly on $\mathcal{H}$, and there exist a unitary W s.th.

$$H_\infty = W H W^*.$$

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Application:

Diagonalization of quadr. Hamilt. by Bog. trafo

$$\mathbb{H} = \sum_{k,l} \left\{ \Omega_{k,l}\, a_k^* a_l + B_{k,l}\, a_k^* a_l^* + \overline{B_{k,l}}\, a_l a_k + C \cdot \mathbb{1} \right\}$$

w.l.o.g.

$$\Omega = \Omega^* \quad \left(\Omega_{k,l} = \overline{\Omega_{l,k}}, \quad B_{k,l} = B_{l,k} \right)$$
$$B = B^T$$

Note: $(B = 0) \iff ([\mathbb{H}, \mathbb{N}] = 0)$

$\Rightarrow$ Choose $A := \mathbb{N}$!

Define

$$\mathbb{H}_t = \sum_{k,l} \left\{ \Omega_t(k,l)\, a_k^* a_l + B_t(k,l)\, a_k^* a_l^* + h.c. \right\} + C_t \cdot \mathbb{1}$$

with $\dot{\mathbb{H}}_t = [\mathbb{H}_t, [\mathbb{H}_t, \mathbb{N}]]$, $\mathbb{H}_0 = \mathbb{H}$.

$$\iff \begin{cases} \dot{\Omega}_t = -16\, B_t B_t^*, & \Omega_0 := \Omega \\ \dot{B}_t = -2\left(\Omega_t B_t + B_t \Omega_t^T\right), & B_0 := B \\ C_t = C_0 + 8\int_0^t \|B_s\|_{HS}^2\, ds. & \end{cases} \quad (2)$$

(A1) $\Omega = \Omega^* \geq 0$,

(A2) $B = B^T \in \mathcal{L}^2(\mathcal{H})$ with $\mathrm{Ran}(B) \subseteq \mathrm{Ran}(\Omega)$,

(A3) $\Omega \geq 4\, B (\Omega^T)^{-1} B^*$,

(A4) $\Omega^{-1} B \in \mathcal{L}^2(\mathcal{H})$, $4\, B(\Omega^T)^{-2} B^* \leq 1$.

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Thm. 4: Assume (A1), (A2), (A3). Then Eq. (2) has a unique solution $(\Omega_t, B_t)_{t\geq 0}$, and there exists a unitary $(\mathbb{U}_t)_{t\geq 0}$ on $\mathcal{F}(\mathcal{H})$ s.th.

$$\mathbb{U}_t \mathbb{H} \mathbb{U}_t^* = \sum_{k,l} \Big\{ \Omega_t(k,l)\, a_k^* a_l + B_t(k,l)\, a_k^* a_l^* \tag{3}$$
$$+ \overline{B_t(k,l)}\, a_k a_l + C_0 + 8\int_0^t \|B_s\|_{HS}^2 ds \Big\},$$

where $\|B_t\|_{HS} \searrow 0$, as $t \to \infty$.

If furthermore (A4) is assumed then $\operatorname*{s-lim}_{t\to\infty} \mathbb{U}_t =: \mathbb{U}_\infty$ exists and is unitary,

$$\mathbb{H}_\infty = \mathbb{U}_\infty \mathbb{H} \mathbb{U}_\infty^* = \sum_{k,l} \Omega_\infty(k,l)\, a_k^* a_l + C_0 + \int_0^\infty \|B_s\|_{HS}^2 ds.$$