LEVEL SET SOLUTION OF AN INVERSE ELECTROMAGNETIC CASTING PROBLEM USING TOPOLOGICAL ANALYSIS

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ABSTRACT
In this communications we present an algorithm to solve an inverse electromagnetic casting problem. We look for a suitable set of electric wires such that the electromagnetic field induced by an alternating current passing through them makes a given mass of liquid metal acquire a predefined shape. The inverse electromagnetic casting problem is formulated as an optimization problem using a shape functional based on the Kohn-Vogelius criterion. The numerical level set method proposed is defined by means of second order topological asymptotic analysis. Numerical examples are presented showing that the proposed technique is effective to design suitable inductors.

1. INTRODUCTION
In metallurgical industry the electromagnetic casting allows contactless heating, shaping and controlling of chemical aggressive hot melts. The quasi-static mathematical model of problem studied here concerns the case of a vertical column of liquid metal falling down into an electromagnetic field created by vertical inductors. In the two-dimensional model, the inverse electromagnetic casting problem consists in finding a distribution of inductors in order that the generated exterior field makes the horizontal cross-section of the molten metal attain a prescribed shape. From a practical point of view, the magnetic field has to be created by inductors which are, each one, a set of bounded insulated strands. In this paper we look for configurations of inductors considering different topologies and shapes. The numerical optimization procedure proposed relies on a level set domain representation and a second order topological asymptotic expansion of a shape functional based on the Kohn-Vogelius criterion [1].

2. THE INVERSE PROBLEM
We denote by $\Omega \subset \mathbb{R}^2$ the exterior of the closed and simply connected domain $\omega$ occupied by the cross-section of the metal column. Under suitable assumptions, the magnetic field $\mathbf{B}$ is equal to $(\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}, 0)$ where the flux function $\phi : \Omega \to \mathbb{R}$ is the solution of:

$$
\begin{cases}
-\Delta \phi = \mu_0 j_0 & \text{in } \Omega, \\
\phi = 0 & \text{on } \Gamma = \partial \omega, \\
\phi(x) = c + o(1) & \text{as } \|x\| \to \infty,
\end{cases}
$$

(1)

where the constant $c$ is also an unknown of (1), $\mu_0$ is the vacuum permeability and $j_0$ is the vertical coordinate of the current density vector. We assume that $j_0$ has a compact support in $\Omega$, and that the total current is zero, see [2].

The surface $\Gamma$ in equilibrium is characterized by the following equation:

$$
\frac{1}{2\mu_0} \left| \frac{\partial \phi}{\partial n} \right|^2 + \sigma C = p_0 \quad \text{on } \Gamma,
$$

(2)

where $C$ is the curvature of $\Gamma$ seen from the metal, $\sigma$ is the surface tension of the liquid and the constant $p_0$ is an unknown of the problem. Physically, $p_0$ is the difference between the internal and external pressures.

The existence of solutions of the inverse problem has been analyzed in [3] where the authors stated that, in the zero total current case, the condition $p_0 = \max \sigma C$ must be satisfied to allow the existence of solutions. Denoting $\bar{p} = \sqrt{2\mu_0(p_0 - \sigma C)}$, the solution of (1) must verify $\frac{\partial \phi}{\partial n} = \kappa \bar{p}$ on $\Gamma$, where $\kappa = \pm 1$, with the sign changes located where $C$ reaches its global maximum.

In order to compute $j_0$ let us introduce a shape functional based on the Kohn–Vogelius criterion, namely

$$
\psi(0) = J(\phi) = \frac{1}{2} \|\phi\|_{L^2(\Gamma)}^2 = \frac{1}{2} \int_\Gamma \phi^2 \, ds,
$$

(3)
where the auxiliary function \( \phi \) depends implicitly on \( j_0 \) and \( c \) by solving the following boundary-value problem

\[
\begin{align*}
-\Delta \phi &= \mu_0 j_0 \quad \text{in } \Omega, \\
\frac{\partial \phi}{\partial n} &= \kappa \bar{p} - d(j_0) \quad \text{on } \Gamma, \\
\phi(x) &= c + o(1) \quad \text{as } ||x|| \to \infty,
\end{align*}
\]

where, denoting \(|\Gamma| = \int_\Gamma ds\), \( d(j_0) \) is defined as

\[
d(j_0) = |\Gamma|^{-1} \int_\Omega \mu_0 j_0 dx.
\]

To compute the topological expansion of \( \psi \) we consider a perturbation \( j_\varepsilon \) of \( j_0 \) which is identical to \( j_0 \) everywhere in \( \Omega \) except in a small ball \( B_\varepsilon(\hat{x}) \subset \Omega \) of radius \( \varepsilon \) and center \( \hat{x} \) where the current density is \( \alpha I \).

It can be seen that the second order asymptotic expansion of \( \psi(\varepsilon) = J(\phi) \) is:

\[
\psi(\varepsilon) = \psi(0) + \left( \alpha I \int_\Gamma \phi f ds \right) \pi \varepsilon^2
+ \left( \frac{1}{2} I^2 \int_\Gamma f^2 ds \right) \pi^2 \varepsilon^4,
\]

where \( f \) is the solution of the following exterior boundary-value problem

\[
\begin{align*}
-\Delta f &= (\pi \varepsilon^2)^{-1} \chi_{B_\varepsilon(\hat{x})} \quad \text{in } \Omega, \\
\frac{\partial f}{\partial n} &= -|\Gamma|^{-1} \quad \text{on } \Gamma, \\
\int_\Gamma f ds &= 0.
\end{align*}
\]

The numerical method proposed here is derived following the ideas presented in [4]. Using a level set method we decompose \( \Omega \) into disjoint parts \( \Omega^+, \Omega^- \), and \( \Omega^0 \), representing the regions of \( \Omega \) where the current density \( j_0 \) is, respectively, positive, negative or zero. Let us assume that \( \Omega^+, \Omega^- \), and \( \Omega^0 \) are open domains and that \( \Omega^+ \) and \( \Omega^- \) are bounded. The solution \( j_0 \) of the inverse problem is obtained by an iterative scheme that takes into account the topological expansion \([0]\) and, at each iteration, modifies the domains \( \Omega^+, \Omega^- \), and \( \Omega^0 \) in such a way that the value of the shape functional is reduced.

3. NUMERICAL RESULT

In Fig. 1 the target shape considered is depicted by the dashed line. The current density is \( I = 0.2 \) and \( \sigma = 1.0 \times 10^{-4} \). The radius of the inductors considered is \( \varepsilon = 0.02 \) in normalized unities. The black and gray areas in the figure represent, respectively, the regions occupied by the positive inductors and the negative inductors. The thin solid line is the boundary of the equilibrium shape for the inductors obtained.

4. CONCLUSIONS

The result obtained shows that the optimization algorithm is effective to design the inductors of an electromagnetic casting application. In addition, the algorithm proposed has the advantage of looking for an economic design, since the topological analysis allows the algorithm to locate the inductors where they produce the largest influence on the shape functional.

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6. REFERENCES