AMPLITUDE INDEPENDENT FREQUENCY SYNCHRONISER 
VIA NORMAL FORMS

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ABSTRACT
The problem of local linearizability of the planar linear center nonlinearly perturbed is far from being solved even for low degree nonlinearity \( n \geq 3 \). Synchronization problem [1, 2] consists in bringing appropriate modifications on a given system to obtain a desired dynamic. The desired phase portrait along this contribution contains a compact region around a singular point at the origin in which lie periodic orbits with the same period (independently from the chosen initial conditions). We aim to present the methodology (involving polar normal forms and Gröbner Basis) for tackling this problem using a nonlinear monomial state feedback. As an example, we overcome the challenge for a cubic system. This contribution can be seen as a direct continuation of several new works concerned with the hinting of linearizability conditions in particular [7, 5, 3, 4], it can be also considered as an adaptation of a qualitative theory method to a synchronization problem.

1. INTRODUCTION
We consider the planar dynamical system,
\[
\frac{dx}{dt} = x = X(x, y), \quad \frac{dy}{dt} = y = Y(x, y),
\]
where \((x, y)\) belongs to an open connected subset \( U \subset \mathbb{R}^2 \), \( X, Y \in C^k(U, \mathbb{R}) \), and \( k \geq 1 \). Due to Poincaré : an isolated singular point \( p \in U \) of (1) is a center if and only if there exists a punctured neighborhood \( V \subset U \) of \( p \) such that every orbit in \( V \) is a cycle surrounding \( p \). A center is said to be isochronous if all the orbits surrounding it have the same period. An overview of J.Chavarriga and M.Sabatini [6] present the methods and basic results concerning the problem of the isochronicity, see also [7, 10, 11, 12, 13].

Synchronization problem consists in bringing appropriate modifications on a given system to obtain a desired dynamic, see [1, 2]. Along this paper, the desired phase portrait contains a compact region around a singular point at the origin in which lies periodic orbits with the same period (independently from the chosen initial conditions which is not always the case). More concretely, in this paper we consider the following problem: Starting from a non isochronous polynomial planar system, are there any polynomial perturbation which insures the local linearisability of the perturbed system. In this paper we adopt the normal forms method often used in qualitative theory investigations; center-focus problem, bifurcation problem and local linearisability problem. The problem of local linearisability conditions of of the planar linear center perturbed by cubic nonlinearities (in all generalities on the system parameters 14 parameters) is far to be solved.

In this paper, starting from a 5-parameters non isochronous Chouikha cubic system [7], we identify all possible monomial perturbations of degree \( d \in \{2, 3\} \) insuring local linearisability of the perturbed system. Investigations are based on the normal forms Theory. In the following system as well as in all other considered systems, all parameters are reals.

Consider the real Liénard Type equation
\[
\ddot{x} + f(x)x^2 + g(x) = 0
\]
or equivalently its associated two dimensional (planar) system
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -g(x) - f(x)y^2
\end{align*}
\]
The study of isochronicity of [2] was established first in M. Sabatini paper [10]. The sufficient conditions of the isochronicity of the origin \( O \) for system [3] with \( f \) and \( g \) of class \( C^1 \) are given. In the analytic case, the necessary and sufficient conditions for isochronicity are given by A.R.Chouikha in [7]. In the same paper, the author implemented a new algorithmic method for computing isochronicity conditions for system [3] called C-
algorithm. As an application of this algorithm, the
author studied the following cubic system
\[
\begin{align*}
\dot{x} &= -y + a_{1,2,1}x^2y \\
\dot{y} &= x + a_{2,2,0}x^2 + a_{2,0,2}y^2 + a_{2,3,0}x^3 + a_{2,1,2}xy^2
\end{align*}
\] (4)

where all the parameters values for which system (4)
has an isochronous center at the origin are established
in the following theorem.

We note that the coefficient \(a_{i,j,k}\) denotes the para-
ter of the monomial perturbation of the \(i\)th equa-
tion of the linear isochronous center \((\dot{x} = -y, \dot{y} = x)\)
of degree \(j\) in \(x\) and of degree \(k\) in \(y\).

A 1-parameter perturbation of system (4) is stud-
ied in [8]. Namely, the monomial perturbation of coef-
icient \(a_{1,1,1}\) but the perturbed system stills reducible
to the Liénard type equation for which C-algorithm is
applicable, see [7, 5, 3, 4].

This contribution is devoted to recall the head lines
of the methodology of the Normal Forms algorithm
and to describe how to tackle the synchronized control
problem, namely, how to construct a state feedback
\(\Psi_1\) or \(\Psi_2\) non zero monomial \((\Psi_1\Psi_2 = 0)\) of degree
d \(d \in \{2, 3\}\) such that
\[
\begin{align*}
\dot{x} &= -y + a_{1,2,1}x^2y + \Psi_1(x, y) \\
\dot{y} &= x + a_{2,2,0}x^2 + a_{2,0,2}y^2 + a_{2,3,0}x^3 + a_{2,1,2}xy^2 + \Psi_2(x, y)
\end{align*}
\] (5)
obey to the desired dynamic.

The problem turns to study eight polynomial cubic
systems which are not reducible by the transformations
described in [5] to Liénard type equation. For each
system, we identify the values of the parameters for
which the singular point at the origin is an isochronous
center. Hence it is done for [5].

2. CONCLUSIONS

A further expanded version of this contribution can be
found in [11]. In this contribution we exploit the fact
that any algebraic variety is finitely generated. We
solve the vanishing of the first polynomials (in the sys-
tem parameters) coming from the radial and the an-
gular components of the polar normal form associated
to the studied system (which give the necessary condi-
tions) then we proof the sufficiency by different tech-
nics.

This underline an interesting problem that will be
considered in the author future studies, namely the
depth of the linearizability problem that is: how much
polynomial we must compute to insure that the neces-
sary conditions are also sufficient.

3. ACKNOWLEDGEMENTS

Financial support by the Institut Polytechnique des
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