A BELLMAN APPROACH FOR OPTIMAL CONTROL PROBLEMS ON MULTI-DOMAINS

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ABSTRACT

This article is the starting point of a series of works whose aim is the study of deterministic control problems where the dynamic and the running cost can be completely different in two (or more) complementary domains of the space $\mathbb{R}^N$. The main difficulty is that these dynamic and running cost present discontinuities at the boundary of these domains. We address these questions by using a Bellman approach: our aim is to investigate how to define properly the value function, to deduce what is the right Bellman Equation associated to this problem (in particular what are the conditions on the set where the dynamic and running cost are discontinuous) and to study the uniqueness properties for this Bellman problem. In this first work, we provide a complete answer to these questions in the case of a simple geometry, namely when the two domains are half spaces: we properly define the control problem, identify the different conditions on the hyperplane where the dynamic and the running cost are discontinuous and discuss the uniqueness properties of the Bellman problem by either providing explicitly the minimal and maximal solution or by explicit conditions to have uniqueness.

1. SETTING THE PROBLEM

We consider infinite horizon control problems where we have different dynamic $(\mathbf{b}_i, i = 1, 2)$ and running cost $(l_i, i = 1, 2)$ in the half-spaces $\Omega_1 := \{x \in \mathbb{R}^N : x_N > 0\}$ and $\Omega_2 := \{x \in \mathbb{R}^N : x_N < 0\}$. On each domain $\Omega_i$, we assume that the dynamics and the running cost satisfy standard assumptions: $\mathbf{b}_i$ are uniformly Lipschitz and $\mathbf{b}_i, l_i$ continuous and bounded. We suppose also that the system is controllable on both sides.

In order to define the controlled dynamic and in particular for trajectories which may stay for a while on the hyperplane $\mathcal{H} := \Omega_1 \cap \Omega_2 = \{x \in \mathbb{R}^N : x_N = 0\}$, we follow the pioneering work of Fillippov [7] and use the approach through differential inclusions. As a consequence, we see that, if $\mathcal{A}_i$ is the space of admissible controls in $\Omega_i$, $i = 1, 2$, we have trajectories which stay on $\mathcal{H}$ and which are build through dynamic of the form $\mathbf{b}(x, \alpha_1, \alpha_2, \mu) := \mu \mathbf{b}_1(x, \alpha_1) + (1 - \mu) \mathbf{b}_2(x, \alpha_2)$, with $\mu \in [0, 1]$, $\alpha_i \in \mathcal{A}_i$ and $\mathbf{b}(x, \alpha_1, \alpha_2, \mu) \cdot e_N = 0$ with $e_N := (0, \cdots, 0, 1)$. We denote by $\mathcal{A}_0$ the set of those $(\alpha_1, \alpha_2, \mu)$. The associated cost is $l(x, \alpha_1, \alpha_2, \mu) := \mu l_1(x, \alpha_1) + (1 - \mu) l_2(x, \alpha_2)$ if $l_i$ is the running cost in $\Omega_i$. Once this is done, we can define the value-function and look for the Bellman problem. It is well-known that, for classical infinite horizon problems, i.e. here in $\Omega_1$ and $\Omega_2$, the equations can be written as

$$
\begin{align*}
H_1(x, u, Du) = 0 & \quad \text{in } \Omega_1 \\
H_2(x, u, Du) = 0 & \quad \text{in } \Omega_2
\end{align*}
$$

(1)

where $H_1, H_2$ are the classical Hamiltonians

$$
H_i(x, u, p) := \sup_{\alpha_i \in \mathcal{A}_i} \{-\mathbf{b}_i(x, \alpha_i) \cdot p + \lambda u - l_i(x, \alpha_i)\},
$$

(2)

with $\lambda > 0$ being the actualization factor. From the viscosity solutions’ theory for optimal control problems with state constraints ([9], [3], [2]), it is natural to think that we have to complement these equations by

$$
\begin{align*}
\min\{H_1(x, u, Du), H_2(x, u, Du)\} & \leq 0 & \text{on } \mathcal{H}, \\
\max\{H_1(x, u, Du), H_2(x, u, Du)\} & \geq 0 & \text{on } \mathcal{H}.
\end{align*}
$$

(3)

(4)

This is actually true since the value function naturally satisfies such inequalities. The main interesting questions are now:

1) Does problem \([1], [9], [3], [2] \) have a unique solution ?
2) Does the value function satisfy other properties on $\mathcal{H}$?

3) Do these extra properties allow to characterize the value function as the unique solution of a Bellman problem?

2. OUR RESULTS

Our results give complete answers to the above questions.

**Answer to 1):** Unfortunately we do not have uniqueness for the problem [1]–[3]–[4] but we can identify the maximal subsolution (and solution) and the minimal supersolution (and solution) of [1]–[3]–[4] as value functions of suitable controls problems in the following way: we call “singular” a dynamic $b(x,\alpha_1,\alpha_2,\mu)$ on $\mathcal{H}$ when $b_1(x,\alpha_1)\cdot e_N > 0$ and $b_2(x,\alpha_2)\cdot e_N < 0$ while the non-singular are those for which the $b(x,\alpha_1)\cdot e_N$ have the opposite (may be non strict) signs. The minimal solution is obtained when allowing all kind of controlled strategies (with singular and non-singular dynamic) while the maximal solution is obtained by forbidding singular dynamic. The uniqueness problem comes from the fact that, in some sense, the singular strategies are not encoded in the equations [1]–[3]–[4], while it is the case for the non-singular ones.

**Answer to 2):** If we allow any kind of controlled strategies, both the non-singular and the singular ones, the associated value function also satisfies the inequality

$$H_T(x,u,Du) \leq 0 \quad \text{in } \mathcal{H}, \quad (5)$$

where $H_T(x,u,p)$ is given by

$$\sup_{A_0} \{-b(x,\alpha_1,\alpha_2,\mu)\cdot p + \lambda u - l(x,\alpha_1,\alpha_2,\mu)\}.$$

We emphasize the fact that this viscosity inequality is actually a $\mathbb{R}^N$–1 viscosity inequality (meaning that we are considering maximum point relatively to $\mathcal{H}$ and not to $\mathbb{R}^N$); it reflects the suboptimality of the controlled trajectories which stays on $\mathcal{H}$.

**Answer to 3):** There exists a unique solution of [1]–[3]–[4]–[5]. In other words, the uniqueness gap for [1]–[3]–[4] just comes from the fact that a subsolution of [1]–[3]–[4] does not necessarily satisfy [5] and this is due to the difficulty to take into account (at the equation level) some singular strategies. We illustrate this fact by an explicit example in dimension 1.

We end by remarking that there are rather few articles on the same topic, at least if we insist on having such structure with a discontinuous dynamic. We cite the work of [6] that consider a similar method to construct a numerical method for a calculus of variation problem with discontinuous integrand or [4] that study an optimal control problem on stratified domains. Problems with a discontinuous running cost were addressed by either Soravia [10] or Camilli & Siconolfi [3] (even in an $L_\infty$-framework). We finally remark that problems on network (see [1]–[8]) share the same kind of difficulties.

3. REFERENCES


