Identification of a time-dependent point source in a linear transport equation with spatially varying coefficients: detection of pollution source

Adel Hamdi and Imed Mahfoudhi

Laboratoire de mathématiques LMI
Institut National des Sciences Appliquées de Rouen
Avenue de l’Université, 76801 Saint-Etienne-du-Rouvray Cedex-France
Email: Adel.Hamdi@insa-rouen.fr, Imed.Mahfoudhi@insa-rouen.fr

November 14, 2011

We are interested in the identification of a time-dependent point source occurring in the right-hand side of a one-dimensional evolution linear advection-dispersion-reaction equation. The originality of this study consists in considering the nonlinear inverse source problem in the general case of transport equations with spatially varying diffusion, velocity and reaction coefficients.

One motivation for our study concerns an environmental application that regards the identification of pollution sources in surface water: in a river, for example, the introduction of organic matter which could have as origin city sewages, industrial wastes,... usually drops to too low the level of the dissolved oxygen in the water. Since the lack of dissolved oxygen represents a serious threat to the diversity of the aquatic life, then localizing pollution sources and recovering the history of the loaded organic matter could play a crucial role in preventing worse consequences regarding the perish of many aquatic species as well as in alerting downstream drinking water stations about the presence of an accidental pollution. This can be done by recording the BOD (Biological Oxygen Demand) concentration which represents the amount of dissolved oxygen required by the micro-organism living in the river to decompose the introduced organic substances [1]. Therefore, the more organic material there is, the higher the BOD concentration. Besides, in a portion of a river represented by a segment $(0, \ell)$ which we suppose controlling during a time $T$, the BOD concentration denoted here by $u$ is governed in $Q = (0, \ell) \times (0, T)$ by the following one-dimensional parabolic partial differential equation appended to initial and boundary conditions, for more details see [3, 5]:

\[
\begin{align*}
\partial_t u(x,t) - \partial_x \left(D(x)\partial_x u(x,t) - V(x)u(x,t)\right) + R(x)u(x,t) &= F(x,t) \quad \text{in } Q \\
u(x,0) &= 0 \quad \text{for } 0 < x < \ell \\
u(0,t) = \partial_x u(\ell,t) &= 0 \quad \text{for } 0 < t < T
\end{align*}
\]
where $F(x, t) = \lambda(t)\delta(x - S)$ represents the pollution source with $S$ is the source location and $\lambda \in L^2(0, T)$ is its time-dependent intensity function. Here, $\delta$ is the Dirac mass whereas $V$, $D$ and $R$ are respectively the flow velocity, the diffusion and the reaction coefficients. In addition, $R$ is a continuous function on $[0, \ell]$, $V$ is a function of $C^1$-class on $[0, \ell]$ and $D$ is a twice piecewise continuously differentiable function on $[0, \ell]$ such that $D(x) > 0$ for all $x \in [0, \ell]$.

Note that due to the linearity and in view of the superposition principle, the use of a non-zero initial condition and/or inhomogeneous boundary conditions do not affect the results established in this study. Furthermore, it is well known that the problem (1) admits a unique solution $u$ smooth enough to use its value at any point $(x, t) \in Q$, see [4]. Therefore, given two observation points $a$ and $b$ of $(0, \ell)$, we can define the observation operator $M[F] := \{u(a, t), u(b, t); \quad 0 < t < T\}$. This is the so-called direct problem.

The nonlinear inverse source problem with which we are concerned here is: assuming available the records $\{d_a(t), d_b(t); \quad 0 < t < T\}$ of the concentration $u$ at the two observation points $a$ and $b$, find the elements $S$ and $\lambda$ defining the sought time-dependent point source $F(x, t) = \lambda(t)\delta(x - S)$ such that $M[F] = \{d_a(t), d_b(t); \quad 0 < t < T\}$.

**Obtained results:** We prove a main condition on the spatially varying flow characteristics $V$, $D$ and $R$ that implies the identifiability of the elements $S$ and $\lambda$ defining the sought time-dependent point source $F(x, t) = \lambda(t)\delta(x - S)$ from the records $\{d_a(t), d_b(t); \quad 0 < t < T\}$. Note that in the case of constant coefficients $V$, $D$ and $R$, this main condition is equivalent to have positive the discriminant of the characteristic equation associated to the operator that is the adjoint of the spatial part of the operator defined by the first equation in (1). Having this discriminant positive is an essential ingredient in the identifiability theorem proved in [2]. Therefore, the results of the present study seem to be a generalization of those found for the model with constant coefficients.

Then, we establish a quasi-explicit identification procedure that determines the source location $S$ as the zero of a continuous and strictly monotonic function. Once $S$ is known, we recover the time-dependent intensity function $\lambda$ from solving a deconvolution problem.

**References**


