INSENSITIZING CONTROL FOR THE NAVIER-STOKES EQUATIONS

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ABSTRACT

In this paper, we deal with the existence of insensitizing controls for the Navier-Stokes equations, more precisely, controls insensitizing the $L^2$-norm of the observation of the solution in an open subset $\mathcal{O}$ of the domain, under suitable assumptions on the data. This problem is equivalent to an exact controllability result for a cascade system. First we prove a global Carleman inequality for the linearized Navier-Stokes system with right-hand side, which leads to the null controllability at any time $T > 0$. Then, we deduce a local null controllability result for the cascade system.

1. INTRODUCTION

The Navier-Stokes equations describe the motion of an incompressible fluid such as water, air, oil...

Let $\Omega \subset \mathbb{R}^N$ ($N = 2$ or $3$) be a bounded connected open set whose boundary $\partial \Omega$ is smooth enough (for instance of class $C^2$). Let $\omega$ and $\mathcal{O}$ be two subsets of $\Omega$ (resp. the control domain and the observatory) such that $\omega \cap \mathcal{O} \neq \emptyset$ and let $T > 0$. Set $Q = \Omega \times (0, T)$ and $\Sigma = \partial \Omega \times (0, T)$. We are interested in the system

\[
\begin{cases}
y_t - \Delta y + (y, \nabla)y + \nabla p = f + v_1 \omega & \text{in } Q, \\
\nabla \cdot y = 0 & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y|_{t=0} = y^0 + \tau y^0 & \text{in } \Omega.
\end{cases}
\]

Here, $y(x, t) = (y_i(x, t))_{1 \leq i \leq N}$ is the velocity of the particles of an incompressible fluid, $v$ is a distributed control localized in $\omega$, $f(x, t) = (f_i(x, t))_{1 \leq i \leq N} \in L^2(Q)^N$ is a given, externally applied force, and the initial state $y|_{t=0}$ is partially unknown. We only suppose that $y|_{t=0}$ is divergence free, $\|y^0\|_{L^2(\Omega)^N} = 1$ and that $\tau$ is a small unknown real number.

The aim of this paper is to prove the existence of controls that insensitize some functional $J_\tau$ (the sentinel) depending on the velocity field $y$. That is to say, we have to find a control $v$ such that the influence of the unknown data $\tau y^0$ is not perceptible for our sentinel:

\[
\frac{\partial J_\tau(y)}{\partial \tau}|_{\tau=0} = 0 \quad \forall y^0 \in B_{L^2(\Omega)^N}(0, 1). \tag{2}
\]

In the pioneering work [1], J.-L. Lions considers this kind of problem and introduces many related questions. One of these questions, in non-classical terms, was the existence of insensitizing controls for the Navier-Stokes equations (see [1], page 56).

Throughout the literature, the usual sentinel is given by the square of a Hilbertian norm of a function of the state variable $y$ (see [2]). Here, we will be interested in insensitizing the following functional

\[
J_\tau(y) = \frac{1}{2} \int_0^T \int_{\mathcal{O} \times (0, T)} |y|^2 \, dx \, dt. \tag{3}
\]

We obtain the following result:

**Theorem 1.1** Let $m > 4$ be a real number and $y^0 = 0$. Then there exists a constant $C_*$ depending on $\omega$, $\Omega$, $\mathcal{O}$ and $T$ such that for any $f \in L^2(Q)^N$ satisfying $\|e^{C_*/t^m} f\|_{L^2(Q)^N} < +\infty$, there exists insensitizing controls $v$ of the functional $J_\tau$ given by (2). This result deserves some explanations: all hypotheses seem very restrictive. First it seems natural that insensitizing [3] with respect to small disturbances near the equilibrium is the relevant property; this explains the condition $y_0 = 0$ in the theorem. Also, we do not know whether Theorem 1.1 still holds if $y_0$ is not identically zero (see [3]). Furthermore, in addition to insensitizing the functional $J_\tau$ one can steer the state $y_0$ (solution of (1) for $\tau = 0$ ) to $0$ at time $t = T$ just by paying the prize of an extra condition on $f$ at time $t = T$:

\[
\|e^{C_*/(T-t)^m} f\|_{L^2(Q)^N} < +\infty, \tag{4}
\]

for a constant $C_*$ that is may be different.
As long as insensitizing controls have been considered, the condition \( \omega \cap \partial \Omega \neq \emptyset \) has always been imposed. But, from [4], we see that this is not a necessary condition. For instance, the authors have proved in [4] that there exists \( \epsilon \)-insensitizing controls of \( J_r \) for linear heat equations with no intersecting observation and control regions in one space dimension.

### 2. OUTLINE OF THE PROOF

As we have already mentioned, the special form of the sentinel allows us to reformulate our insensitizing problem as a controllability problem of a cascade system (see [2], for instance). In particular, condition (2) is equivalent to \( z_{|t=0} = 0 \) in \( \Omega \), where \( z \) together with \( w \) solves the following coupled system:

\[
\begin{align*}
    w_t - \Delta w + (w, \nabla)w + \nabla p^0 &= f + v1_{\omega} & \text{in } Q, \\
    \nabla \cdot w &= \nabla \cdot z = 0 & \text{in } Q, \\
    w &= z = 0 & \text{on } \Sigma, \\
    w_{|t=0} = y^0, \quad z_{|t=T} = 0 & \text{in } \Omega.
\end{align*}
\]

Here, \((w, p^0)\) is the solution of system (1) for \( \tau = 0 \). The equation on \( z \) corresponds to a formal adjoint of the equation satisfied by the derivative of \( y \) with respect to \( \tau \) at \( \tau = 0 \) and we have denoted

\[
((z, \nabla^i)w)_i = \sum_{j=1}^N z_j \partial_i w_j \quad i = 1, \ldots, N.
\]

As usual, we linearize the system around the equilibrium and we look for a control \( v \in L^2(\omega \times (0, T)) \) such that, under suitable decreasing properties on \( f_1 \) and \( f_2 \), the solution to

\[
\begin{align*}
    w_t - \Delta w + \nabla p &= f_1 + v1_{\omega} & \text{in } Q, \\
    \nabla \cdot w &= \nabla \cdot z = 0 & \text{in } Q, \\
    w &= z = 0 & \text{on } \Sigma, \\
    w_{|t=0} = z_{|t=T} = 0 & \text{in } \Omega,
\end{align*}
\]

satisfies

\[
    z_{|t=0} = 0 \quad \text{in } \Omega.
\]

The proof of the existence of such controls relies on suitable observability estimate (see [5]). In the context of the null controllability analysis of parabolic systems, Carleman estimates (see [6]) are a convenient tool. Carleman estimates are weighted inequalities with degenerate weight depending on the control domain. We derive such an estimate for the adjoint of (6), which, in turn, gives us a suitable decay rate for \( f_1 \) and \( f_2 \). To conclude, the previous result allows us to locally invert a nonlinear operator associated to the nonlinear system

\[
\begin{align*}
    \mathcal{A}(w, z, v) &= (w_t - \Delta w + (w, \nabla)w + \nabla p - v1_{\omega}, \\
    -z_t - \Delta z + \nabla q - w, \nabla)w + \nabla q - w1_{\omega}) & \forall(w, z, p, q, v) \in \mathcal{E},
\end{align*}
\]

for a suitable space \( \mathcal{E} \) which depends on the weight functions of the Carleman estimate.

### 3. CONCLUSIONS

Most known results concerning insensitizing controls are for parabolic systems (see [2], [4], [7]...). Even in this case, if the sentinel no longer has this form it seems very hard to tackle the problem using usual control methods. Furthermore, the question of insensitizing controls with no intersecting observation and control regions remains a challenging issue.

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### 5. REFERENCES


