NUMERICAL SIMULATION OF THE EXACT BOUNDARY CONTROLLABILITY OF THE 1D WAVE EQUATION WITH POTENTIAL

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ABSTRACT
A numerical simulation of the exact boundary controllability of the wave equation with potential is studied. We formulate the question of controllability as a limit of a penalized optimal control problem. We prove the existence and the unicity of a solution for the continuous problem. The approach used for discretization is “first discretize, then optimize”. We then write the optimality discrete system which is resolved by the quasi-Newton method BFGS. We finally show the numerical results that prove the convergence of the penalized control and the approached state respectively to the exact control and the desired state.

1. INTRODUCTION
The problem of exact controllability consists in guiding by means of feasible control, the solution of a system governed by a partial differential equation to a desired final configuration. Many works on numerical simulation of the exact controllability (distributed and boundary) of the wave equation have been treated (2, 3, 7). We are interested in this paper to the numerical study of the exact boundary controllability of the 1D wave equation with stationary potential. The applications are very varied: we cite for example, the propagation of an elastic wave on a non null stiffness string (6).

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + q(x)u & = 0, \quad \text{in } ]0, L[ \times ]0, T[,

u(0, t) = f(t), \quad u(L, t) = 0, \quad \text{in } ]0, T[,

u(x, 0) = 0, \quad \partial_t u(x, 0) = 0, \quad \text{in } ]0, L[.
\end{aligned}
\]

Le potential \( q \) modelizes the stiffness of the string and \( f(t) \) the source or the control.

Given a function \( v \in L^2([0, L]) \), it has been proved in (3) that there exists a source \( f \in L^2([0, T]) \) such that \( u_f(x, T) = v(x) \) where \( u_f \) is the solution of the previous problem.

Following the idea of J-L Lions (5), we formulate the question of exact controllability as a limit of a penalized optimal control problem.

\[
\min_{f \in L^2([0, T])} J(u_f, f)
\]

The function \( J \) is given by:

\[
J(u_f, f) = \frac{1}{2\varepsilon} \int_0^L (u_f(x, T) - v(x))^2 \, dx + \frac{1}{2} \int_0^T f^2(t) \, dt
\]

\( \varepsilon \) is the penalization parameter which tends to zero and the energy term \( \frac{1}{2} \int_0^T f^2(t) \, dt \) is the Tychonoff regularization.

We prove the existence and the unicity of an optimal control by using classical optimization techniques and we characterize it by an optimality system using the adjoint state method.

The numerical approach followed here is “first discretize, then optimize”, which consists in: first discretizing the Lagrangian associated to the functional \( J \) and then writing the discrete optimality conditions. The method used for discretizing the state equation and the adjoint state equation is the finite difference method \( \theta \)-schema which is energy conservative, time reversible and stable under a CFL condition. The optimization method is the quasi-Newton method BFGS which is convergent even for small values of \( \varepsilon \).

2. NUMERICAL RESULTS
We present the numerical results that show the convergence of the penalized control and the approached state respectively to the exact control and the desired
state when the penalization parameter tends to zero. In figure (2), the desired state (gaussian beam) is identical to the computed one with \( \varepsilon = 10^{-3} \).

3. CONCLUSION

We propose in this paper a robust method for computing the exact control. Meanwhile, the method is not efficient for real-time applications.

4. REFERENCES


Figure 1: Dirac potential.

Figure 2: Computed state (\( \varepsilon=1 \)) and desired state.

Figure 3: Optimal control.