Robust Fuzzy Observer for a Harvested Fish Population Model

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Abstract—This paper deals with the problem of observer (state estimation) of nonlinear uncertain systems to study the continuous age structured model of a harvested fish population, in order to get an estimation of the biomass of fishes by age class. In our case the fishing effort is considered as a control term, the age classes as a states and the quantity of captured fish as a measured output. The uncertain non-linear model is first represented by a Takagi-Sugeno multimodel. Next, we develop a technique for designing a multimodel observer (called also software sensor) corresponding to this system and show its asymptotic convergence. The design conditions are given in linear matrix inequalities (LMI) terms which can be solved efficiently using existing numerical tools. The simulation results demonstrate the effectiveness of the proposed method.

1. Takagi-Sugeno Fuzzy Model
1.1. Model Representation

A dynamic T-S fuzzy model is described by a set of fuzzy “IF … THEN” rules with fuzzy sets in the antecedents and dynamic linear time-invariant systems in the consequents. A generic T-S plant rule can be written as follows [10]:

Model Rule i:

\[ \text{IF } z_i(t) \text{ is } M_{i_1} \text{ and } \ldots \text{ and } z_p(t) \text{ is } M_{i_p} \]

\[ \text{THEN } \] 

\[ \dot{x}(t) = A_{i}x(t) + B_{i}u(t) ; \quad y(t) = C_{i}x(t) \]

Here, \( M_{i} \) is the fuzzy set and \( r \) is the number of model rules; \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the input vector, \( y(t) \in \mathbb{R}^q \) is the output vector, \( A_{i} \in \mathbb{R}^{n \times n}, B_{i} \in \mathbb{R}^{n \times m}, \) and \( C_{i} \in \mathbb{R}^{q \times n} ; z_i(t), \ldots, z_p(t) \) are known premise variables that may be functions of the state variables, external disturbances, and/or time.

We will use \( z(t) \) to denote the vector containing all the individual elements \( z_1(t), \ldots, z_p(t) \).

Given a pair of \((x(t),u(t))\), and using singleton fuzzifier, max-product inference and center average defuzzifier, we can write the aggregated fuzzy model as:

\[ \hat{x}(t) = \sum_{i=1}^{r} \mu_i(z(t)) \left( A_i x(t) + B_i u(t) \right) \] (1)

where:

\[ \sum_{i=1}^{r} \mu_i(z(t)) = 1 \quad \text{and} \quad \mu_i(z(t)) \geq 0, \quad i = 1, 2, \ldots, r, \text{ for all } t. \]

The global output of T-S model is interpolated as follows:

\[ y(t) = \sum_{i=1}^{r} \mu_i(z(t)) C_i x(t) \] (2)

1.2. Robust fuzzy observer design

Let us consider the following uncertain multiple model with unmeasurable decision variables:

\[ \dot{x}(t) = \sum_{i=1}^{r} \mu_i(z(t))(A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t) \] (3)

\[ y = Cx + Du \]

Where:

\[ \Delta A_i(t) = M^A A_i(t) N^A_i \quad \text{and} \quad \Delta B_i(t) = M^B B_i(t) N^B_i \] (4)

With

\[ \Sigma^A_i(t) \Sigma^A_i(t) < I, \forall t \quad \text{and} \quad \Sigma^B_i(t) \Sigma^B_i(t) < I, \forall t \]

where I is the identity matrix and \( o(t) \) is a bounded measurement noise.

The T-S observer for the T-S model (1) is written as follows:

\[ \hat{x}(t) = \sum_{i=1}^{r} \mu_i(z(t))(A_i \hat{x}(t) + B_i u(t) + G_i(y(t) - \hat{y}(t))) \] (5a)

\[ \dot{\hat{y}}(t) = C \hat{x}(t) \] (5b)

Let us define the state estimation error:

\[ e(t) = x(t) - \hat{x}(t) \] (6)

Assumption 1: In this subsection, we suppose that the following hypotheses hold:

• A1. The system (3) is assumed to be stabe.
• A2. The weighting functions $\mu_i(x)$ are Lipschitz:
• A3. The functions $\mu_i(x)x$ are Lipschitz:
• A4. The input $u(t)$ of the system is bounded:

**Theorem 1** [5]: the state estimation error converges asymptotically to zero, and the $L_2$ gain of the transfer from $\tilde{o}$ to $\tilde{e}$ is minimal if there exists positive and symmetric matrices $P$, gains $K_i$, positive scalars $\bar{y}$, $\epsilon_1$, $\epsilon_2$, solution of the following problem:

$$\min_{P,K_i,\epsilon_1,\epsilon_2,\bar{y}} \psi$$

s.t. to the constraints for all $i \in \{1, \ldots, r\}$

$$
\begin{bmatrix}
\Psi_{11i} & P & -K_i D & 0 & 0 & P M_i^A & P M_i^B \\
0 & \Psi_{22i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Psi_{33i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Psi_{44i} & 0 & 0 & 0 \\
(M_i^A)^T P & 0 & 0 & 0 & 0 & -\epsilon_1 I & 0 \\
(M_i^B)^T P & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 I
\end{bmatrix} - \begin{bmatrix}
\Psi_{11i} = A_i^T P + PA_i + C_i^T K_i C + I \\
\Psi_{22i} = -\bar{y} I \\
\Psi_{33i} = -\bar{y} I + \epsilon_1 (N_i^A)^T N_i^A \\
\Psi_{44i} = -\bar{y} I + \epsilon_2 (N_i^B)^T N_i^B
\end{bmatrix} < 0
$$

The gain of the observer are computed from $\tilde{g}_i = P_i^{-1} K_i$ (7)

And attenuation level is derived from $\gamma = \sqrt{\bar{y}}$ (8)

**Proof**: See [5]

2. APPLICATION: stabilization of a fish population system

2.1. Problem Formulation and Assumptions

We consider a population of exploited fish which is structured in $n$ age classes ($n \geq 2$), where every stage $i$ is described by the evolution of its biomass $X_i$ for $0 \leq i \leq n$. Each stage in the stock ($i = 1 \ldots n$) is characterized by its fecundity, mortality and predation rates. In addition, a fishing effort is included in the global mortality term. The dynamic of the fish population can be represented by the following system of ordinary differentials equations [14][15]:

$$
\begin{align*}
\dot{X}_0 &= -\alpha_0 X_0 + \sum_{i=1}^{n} f_i l_i X_i - \sum_{i=0}^{n} p_i X_i X_0 + r \\
\dot{X}_1 &= \alpha X_0 - (\alpha_1 + q_1 E) X_1 \\
\vdots \\
\dot{X}_n &= \alpha X_{n-1} - (\alpha_n + q_n E) X_n \\
Y &= q_1 X_1 + q_2 X_2 + \ldots + q_n X_n
\end{align*}
$$

(9)

Where:

- $\alpha_i = \alpha + M_i$:
  - $M_i$ : is the natural mortality of the individuals of the $i^{th}$ age class;
  - $\alpha$ : is the linear aging coefficient;
  - $p_i$ : is the predation parameter of class $i$ on class $0$;
  - $f_i$ : is the fecundity rate of class $i$;
  - $l_i$ : is the reproduction efficiency of class $i$;
  - $q_i$ : is the catchability of the individuals of the $i^{th}$ age class;
  - $X_i$ : is the biomass of class $i$;
  - $E$ : is the fishing effort at time $t$ and is regarded as an input;
  - $Y$ : is the total catch per unit of effort and is regarded as output;
  - $r$ : can be viewed as an unknown signal composed of model uncertainties, non-linearities and disturbances, etc.; which affects the other states of the system.

2.2. Construction of Fuzzy Model

For simplicity, we consider that $n=3$, and $x_i \in [-a, a], a \in \mathbb{R}^+$. One considers here a population with three stages age ($n=2$), and $x_i \in [-a, a], a \in \mathbb{R}^+$. Then, the nonlinear system (9) is represented by the following fuzzy model.
3. Simulation Results and Discussion

To visualize the fuzzy observer obtained from the proposed method, we consider a numerical example obtained from a fishery characterized by the parameter values given in Table 1 which are retained from the literature [14]. Here we have employed for the simulation a constant fishing effort $E(t) = \bar{E}$ and arbitrary initial states: $X(0) = (1.9; 3.4; 5)$ and $\hat{X}(0) = (0.9; 1.1; -0.6)$. 

\[
A_1 = \begin{bmatrix} k_0 + p_0a & k_1 + p_1a & k_2 + p_2a \\ \alpha & -(a_1 + q_1\bar{E}) & 0 \\ 0 & \alpha & -(a_2 + q_2\bar{E}) \end{bmatrix}; \quad B_1 = \begin{bmatrix} 0 \\ -q_1X_1^t - q_1a \\ -q_2X_2^t - q_2a \end{bmatrix}
\]

\[
A_2 = \begin{bmatrix} k_0 + p_0a & k_1 + p_1a & k_2 + p_2a \\ \alpha & -(a_1 + q_1\bar{E}) & 0 \\ 0 & \alpha & -(a_2 + q_2\bar{E}) \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0 \\ -q_1X_1^t - q_1a \\ -q_2X_2^t + q_2a \end{bmatrix}
\]

\[
A_3 = \begin{bmatrix} k_0 + p_0a & k_1 + p_1a & k_2 + p_2a \\ \alpha & -(a_1 + q_1\bar{E}) & 0 \\ 0 & \alpha & -(a_2 + q_2\bar{E}) \end{bmatrix}; \quad B_3 = \begin{bmatrix} 0 \\ -q_1X_1^t + q_1a \\ -q_2X_2^t - q_2a \end{bmatrix}
\]

\[
A_4 = \begin{bmatrix} k_0 + p_0a & k_1 + p_1a & k_2 + p_2a \\ \alpha & -(a_1 + q_1\bar{E}) & 0 \\ 0 & \alpha & -(a_2 + q_2\bar{E}) \end{bmatrix}; \quad B_4 = \begin{bmatrix} 0 \\ -q_1X_1^t + q_1a \\ -q_2X_2^t + q_2a \end{bmatrix}
\]

\[
A_5 = \begin{bmatrix} k_0 + p_0a & k_1 + p_1a & k_2 + p_2a \\ \alpha & -(a_1 + q_1\bar{E}) & 0 \\ 0 & \alpha & -(a_2 + q_2\bar{E}) \end{bmatrix}; \quad B_5 = \begin{bmatrix} 0 \\ -q_1X_1^t - q_1a \\ -q_2X_2^t - q_2a \end{bmatrix}
\]

\[
A_6 = \begin{bmatrix} k_0 + p_0a & k_1 + p_1a & k_2 + p_2a \\ \alpha & -(a_1 + q_1\bar{E}) & 0 \\ 0 & \alpha & -(a_2 + q_2\bar{E}) \end{bmatrix}; \quad B_6 = \begin{bmatrix} 0 \\ -q_1X_1^t - q_1a \\ -q_2X_2^t + q_2a \end{bmatrix}
\]

\[
A_7 = \begin{bmatrix} k_0 + p_0a & k_1 + p_1a & k_2 + p_2a \\ \alpha & -(a_1 + q_1\bar{E}) & 0 \\ 0 & \alpha & -(a_2 + q_2\bar{E}) \end{bmatrix}; \quad B_7 = \begin{bmatrix} 0 \\ -q_1X_1^t + q_1a \\ -q_2X_2^t - q_2a \end{bmatrix}
\]

\[
A_8 = \begin{bmatrix} k_0 + p_0a & k_1 + p_1a & k_2 + p_2a \\ \alpha & -(a_1 + q_1\bar{E}) & 0 \\ 0 & \alpha & -(a_2 + q_2\bar{E}) \end{bmatrix}; \quad B_8 = \begin{bmatrix} 0 \\ -q_1X_1^t + q_1a \\ -q_2X_2^t + q_2a \end{bmatrix}
\]

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</table>

Table 1: Parameter values used for simulation

\[
M_i \in [0.1 \ 0.1 \ 0.1]; \quad NiA = [0.1 \ 0.1 \ 0.1]; \quad M_i = [0.1; 0.1; 0.1];
\]

The gains of the fuzzy observer are computed according to (13):

\[
L_1 = \begin{bmatrix} -3.2031 \\ 5.9267 \\ -1.8401 \end{bmatrix}; \quad L_2 = \begin{bmatrix} -3.0336 \\ 6.2745 \\ -2.1286 \end{bmatrix}; \quad L_3 = \begin{bmatrix} -3.0336 \\ 6.2745 \\ -2.1286 \end{bmatrix}; \quad L_4 = \begin{bmatrix} -3.0336 \\ 6.2745 \\ -2.1286 \end{bmatrix}
\]

\[
L_5 = \begin{bmatrix} -3.2031 \\ 5.9267 \\ -1.8401 \end{bmatrix}; \quad L_6 = \begin{bmatrix} -3.0336 \\ 6.2745 \\ -2.1286 \end{bmatrix}; \quad L_7 = \begin{bmatrix} -3.0336 \\ 6.2745 \\ -2.1286 \end{bmatrix}; \quad L_8 = \begin{bmatrix} -3.2031 \\ 5.9267 \\ -1.8401 \end{bmatrix}
\]

$\Sigma_A$ and $\Sigma_B$ are identical and depicted in the figure 1.

The obtained results are shown in figures 2,3 and 4 which give time evolution of the class $X_i$ and their estimate $\hat{X}_i$. The state estimation error converges (see figure 5) and the gain of the transfer from $E(t)$ to $e(t)$ is bounded by $\gamma = 1$. 

\[
\Sigma_A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad \Sigma_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
4. references

1. A. Ait Kaddour, E. H. EL Mazoudi, N. Elalamii, "Static Output Feedback Controller Design Via Takagi-Sugeno Fuzzy Model, Application to a Harvested Fish Population System", V International Conference on Inverse Problems, Control and Shape Optimization (PICOF’10), April 2010, Spain