INVERSE PROBLEM APPROACH TO IDENTIFY MATERIAL PARAMETERS OF AN ELASTIC BODY USING ITS NATURAL FREQUENCIES AND EIGENMODES

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ABSTRACT
Given measurements of the first natural frequencies and eigenmodes (modes of vibration) of a solid tridimensional body, the elastic parameters that characterize the material of the body can be approximated by minimizing appropriate functionals. These functionals measure the residual obtained with the natural frequencies and eigenmodes associated to some values of the material parameters. The main ingredients of the method are the parametric derivatives of the eigenvalues and of the eigenvectors, and the adjoint method. These tools allow the use of gradient type methods to solve the minimization problem. Academic examples using a large dam were performed with successful results contained in [3]. This work is supported by FCT, Financiamento Base 2010, ISFL-1-209, and grant SFRH/BD/75165/2010.

1. INTRODUCTION
Let Ω ⊂ ℝ³ be a solid tridimensional body whose behaviour can be described in the framework of linearized elasticity. The stress-strain constitutive law that models the material of the body is arbitrary (not necessarily isotropic), so it depends on k material parameters s₁, . . . , sₖ:

σ = Cε

where C = C(s₁, . . . , sₖ) is the fourth-order elasticity tensor.

We assume the body is fixed on some part of its boundary Γ_D and is free of surface loads on the rest of the boundary Γ_N. The natural frequencies ωᵢ and eigenmodes uᵢ of the body Ω are respectively the square root of the eigenvalues λᵢ and the eigenvectors of the following problem:

\[
\begin{aligned}
-\text{div}(C \varepsilon(u)) &= \lambda u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma_D, \\
C \varepsilon(u) \cdot n &= 0 \quad \text{on } \Gamma_N.
\end{aligned}
\] (1)

If the first n natural frequencies and eigenmodes can be physically measured in the body with reasonable accuracy (denoted respectively by (\bar{\omega}_i)ᵢ=1,...,n and (\bar{u}_i)ᵢ=1,...,n), the following inverse problem can be posed:

find values for the parameters s₁, . . . , sₖ such that the first n natural frequencies \(\omega_i = \omega_i(s₁, . . . , sₖ)\) and eigenmodes \(u_i = u_i(s₁, . . . , sₖ)\) of problem (1) are close to the corresponding measured natural frequencies \(\bar{\omega}_i\) and eigenmodes \(\bar{u}_i\), for 1 ≤ i ≤ n.

This approach is close to the field of Free Material Optimization, as presented in Chapter 3 of [1].

2. MINIMIZATION PROBLEM
2.1. Definition of the functionals
The following functional is considered to search solutions for the above inverse problem:

\[
J(s) = \sum_{i=1}^{n} |\omega_i(s) - \bar{\omega}_i|^2 + ||u_i(s) - \bar{u}_i||^2_{L^2},
\] (2)

where \(s = (s₁, . . . , sₖ) \in \mathbb{R}^k\) represents the k-tuple containing the values of the material parameters \(s₁, . . . , sₖ\) (the variables of the functional). From the practical point of view, the number of natural frequencies and eigenmodes taken into account should be small (for instance, \(n = 4\)), as only the first few frequencies and modes can be measured with accuracy. We now consider the following parametric minimization problem:

\[
\min_{s \in I} J(s),
\] (3)

where \(I = I₁ \times \ldots \times Iₖ\), and \(I_j = [a_j, b_j]\) is an interval of \(\mathbb{R}\) containing a range of values that has physical meaning for the parameter \(s_j (j = 1, . . . , k)\). The minimization of the functional leads to material parameters corresponding to natural frequencies and eigenmodes \((\omega_i, u_i)ᵢ=1,...,n\) which are close to the corresponding measured data \((\bar{\omega}_i, \bar{u}_i)ᵢ=1,...,n\).
2.2. Derivatives of the natural frequencies and of the eigenmodes

To apply gradient-based methods in problem (3) one needs the derivatives of the natural frequencies and of the eigenmodes with respect to a material parameter $s_j$. Provided differentiability properties of the elasticity tensor $C = C(s)$ with respect to a material parameter $s_j$ and assuming that the eigenvalues of problem (1) are simple, then the eigenvalues and the eigenvectors of (1) are differentiable with respect to $s_j$. The derivative of the eigenvalue $\lambda_i = \lambda_i(s)$ is

$$
\frac{d\lambda_i}{ds_j} = \int_{\Omega} \frac{dC}{ds_j} (s) \epsilon(u_i) \cdot \epsilon(u_i) \, dx,
$$

where the corresponding eigenvector $u_i$ is normalized in $L^2$.

The derivative $\frac{du_i}{ds_j}$ of the eigenvector $u_i = u_i(s)$ is the solution of the following variational problem:

$$
\begin{aligned}
\text{find } & \frac{du_i}{ds_j} \in < u_i >^\perp \text{ such that, for all } v \in V, \\
\int_{\Omega} C(s) \epsilon \left( \frac{du_i}{ds_j} \right) \cdot \epsilon(v) \, dx - \lambda_i \int_{\Omega} \frac{du_i}{ds_j} \cdot v \, dx = 0, \\
\int_{\Omega} \frac{d\lambda_i}{ds_j} u_i \cdot v \, dx - \int_{\Omega} \frac{dC}{ds_j} (s) \epsilon(u_i) \cdot \epsilon(v) \, dx,
\end{aligned}
$$

where $< u_i >^\perp$ denotes the orthogonal complement of the eigenspace generated by $u_i$, with respect to the inner product in $L^2$, and $V$ is the space of test functions:

$$
V = \{ v \in H^1(\Omega)^3 : v_{|_{\Gamma_D}} = 0 \}.
$$

These results were published before in an abstract setting in Banach spaces (see [2]).

The derivative of the functional (2) with respect to $s_j$ can now be computed:

$$
\frac{dJ}{ds_j}(s) = \sum_{i=1}^{n} \frac{\omega_i(s) - \bar{\omega}_i}{\omega_i(s)} \int_{\Omega} \frac{dC}{ds_j} (s) \epsilon(u_i) \cdot \epsilon(u_i) \, dx \\
+ 2 \sum_{i=1}^{n} \int_{\Omega} \frac{du_i}{ds_j} (s) \cdot (u_i(s) - \bar{u}_i) \, dx. \quad (5)
$$

Using this formula requires solving $n \times k$ problems of type (4) for each calculation of the gradient of $J$. The discretization of problem (4) is accomplished with the finite element method, resulting in a large system of linear equations having an indefinite matrix of the form $K - \lambda_i M \in \mathbb{R}^{N \times N}$ and whose resolution is a heavy computational task.

2.3. Adjoint problem and numerical implementation

The adjoint method is used to obtain directly the second term in (5) without the computation of $\frac{du_i}{ds_j}$. For each $i = 1, \ldots, n$ an appropriate adjoint problem is defined:

$$
\begin{aligned}
\text{find } & p_i \in < u_i >^\perp \text{ such that, for all } v \in V, \\
\int_{\Omega} C(s) \epsilon(p_i) \cdot \epsilon(v) \, dx - \lambda_i \int_{\Omega} p_i \cdot v \, dx = 0, \\
-2 \int_{\Omega} \bar{u}_i \cdot v \, dx + 2 \int_{\Omega} \bar{u}_i \cdot u_i(s) \, dx \int_{\Omega} u_i(s) \cdot v \, dx.
\end{aligned}
$$

Once the solutions $p_i$ of the $n$ adjoint problems are known, the second term in (5) is equal to

$$
- \sum_{i=1}^{n} \int_{\Omega} \frac{dC}{ds_j} (s) \epsilon(u_i) \cdot \epsilon(p_i) \, dx. \quad (7)
$$

With the adjoint method, the time required for this computation is only $1/k$ of the time needed if formula (4) is used.

Note that the matrix obtained in the discretization of the adjoint problem (6) is the same as in (4). This is an indefinite matrix for $i \geq 2$. One could apply iterative methods designed for indefinite matrices. However, given the form of the matrix, it is more efficient to use an appropriate decomposition of $\mathbb{R}^n$ in a direct sum. One obtains invariant subspaces to the given matrix, and the restriction to those subspaces becomes positive-definite. The classical methods (such as Conjugate Gradient or Gauss-Seidel) can then be used with the indefinite matrix in order to obtain the component of the solution belonging to the invariant subspace.

The minimization procedure for (2) follows the usual steps, with the values of the material parameters $s$ being successively updated until the stopping criteria is met. The implementation of the BFGS algorithm was done in C++ with aid of open-source libraries for the finite element discretization and for the resolution of the eigenvalue problem.

3. REFERENCES

