ASYMPTOTIC OF THE VELOCITY OF A DILUTE SUSPENSION OF DROPLETS WITH INTERFACIAL TENSION

David Manceau

(joint with Eric Bonnetier and Faouzi Triki)

LMAH, Le Havre, France
Le Havre university
david.manceau@univ-lehavre.fr

ABSTRACT

In this work, we derive the asymptotic expansion of the velocity field of a small deformable droplet immersed in an incompressible Newtonian fluid. Using an appropriate physical scaling of the surface tension with respect to the droplet volume we show that the first order of the asymptotic can be expressed in terms of the velocity field in absence of the droplet and a new kind of moment tensor \[2\]. Moreover, we determine explicitly this tensor for ellipses and ellipsoids.

1. INTRODUCTION

We consider a dilute suspension of deformable droplets in a matrix fluid. We assume the droplet and matrix fluids to be Newtonian and neglect inertia and body forces. The evolution of droplet shape only depends on the viscosity difference between the fluids and on surface tension. The system is thus modeled as a stationary Stokes flow, in a bounded $C^2$-domain $Ω$ of $\mathbb{R}^n$ $(n = 2, 3)$ that contains $k \geq 1$ inhomogeneities $ω^1_ε, \ldots, ω^k_ε$ (the droplets) of small size $ε > 0$. We further assume that the droplets are at a distance $>> ε$ away from each other, and are also away from the boundary, to be able to neglect droplet interactions and their interaction with the boundary $∂Ω$.

We are interested in deriving an asymptotic expansion of the velocity $u_ε$ of the flow as the droplet diameter $ε$ tends to 0. Our goal is to obtain in this manner an approximate model for the flow, that only requires the resolution of a constant coefficient PDE to obtain the velocity $u$ of the homogeneous matrix fluid, and the computation of correction terms that depend on the viscosity ratio, and on the positions, the surface tensions, the geometries of the droplets.

This work is a first step to the determination of the evolution of droplet shape from its moments tensors.

2. STATEMENT OF THE PROBLEM

Let $μ_0 > 0$ be the viscosity of the reference Stokes flow $Ω$ without inhomogeneity. For simplicity, we assume that $Ω$ contains a single droplet

$ω_ε = z_0 + εω$,

of size $ε > 0$, centered at a point $z_0 ∈ Ω$ away from $∂Ω$, i.e. $\text{dist}(z_0, ∂Ω) > 0$. The $C^2$-domain $ω$ contains the origin and represents the rescaled shape of the droplet. We assume that $ω_ε$ contains a Stokes fluid with viscosity $μ_1 ≠ μ_0$. We want to compare the velocity $u_ε$ of the flow to the velocity $u$ of the reference flow in $Ω$ when no droplet is present.

In order to state the equations satisfied by $u$ and $u_ε$, let $g ∈ C^∞(∂Ω)$ such that

$$\int_{∂Ω} g \cdot ν_x \, dσ_x = 0,$$

(1)

where $ν_x$ denotes the outward unit normal to both $∂Ω$ and $∂ω_ε$. Following Tolman’s model [4], we have

$$λ_ε = λ_δ = \tilde{λ}_ε,$$

(3)
where δ is the constant Tolman length. We set

$$
\mu_\varepsilon(x) := \begin{cases} 
\mu_0 & \text{if } x \in \Omega \setminus \Omega_\varepsilon, \\
\mu_1 & \text{if } x \in \partial \Omega_\varepsilon.
\end{cases}
$$

The velocity \( u_\varepsilon \) of the flow is the solution of the following perturbed Stokes problem

$$
\begin{align*}
\text{−Div}(2\mu_\varepsilon c(u_\varepsilon)) + \nabla p_\varepsilon &= \lambda_\varepsilon \varepsilon \nu_x \chi_{\partial \Omega_\varepsilon} \quad \text{in } \Omega, \\
\text{div}(u_\varepsilon) &= 0 \quad \text{in } \Omega, \\
u_\varepsilon &= g \quad \text{on } \partial \Omega.
\end{align*}
$$

where \( \chi_{\partial \Omega_\varepsilon} \) is the characteristic function of \( \partial \Omega_\varepsilon \) and \( \mu_\varepsilon \) is given by (4).

### 3. THE RESULTS

#### 3.1. Asymptotic of the velocity

Let \((G, F) \in H^1(\Omega)^n \times L^2(\Omega)^n\) be the Green tensors associated with the reference system [2] and \(G_i\) be the \(i\)th row of \(G\).

Then, we show [3] that there exists a symmetric fourth order tensor \(V\) and a symmetric matrix \(K\) with zero trace such that, for all \(z \in \Omega\) with \(d := d(z, \Omega_\varepsilon) > 0\) and for all \(i = 1, \ldots, n\), one have

$$
(u_\varepsilon - u)(z)_i = \varepsilon^g e_x(G_i)(z, z_0) \cdot V e_x(u)(z_0) + \varepsilon^g e_x(G_i)(z, z_0) : K + O(\varepsilon^{n+\frac{1}{2}}),
$$

where \(\cdot\) is the scalar product on \(\mathbb{R}^{n \times n}\) and \((u_\varepsilon - u)(z)_i\) is the \(i\)th row of \((u_\varepsilon - u)(z)\).

The rest \(O(\varepsilon^{n+\frac{1}{2}})\) is uniformly bounded by \(c \varepsilon^{n+\frac{1}{2}}\) where \(c\) depends on \(d, \mu_0\) and \(\mu_1\).

The symmetric fourth order tensor \(V\) was first obtained in [1] considering a zero surface tension and is called viscous moment tensor. The additional tensor \(K\) depends on the surface tension and is called curvature moment tensor.

#### 3.2. Curvature moment tensor for ellipses and ellipsoids

In [1], based on layer potentials techniques the authors have derived an analytic expression of the viscous moment tensor \(V\) in the case where \(\omega = \omega_\varepsilon = z_0 + \varepsilon \omega\) is an ellipse or ellipsoid. Using the same techniques, we determine the curvature moment tensor \(K\) for ellipses and ellipsoids.

Assume \(\omega \in \mathbb{R}^2\) is an ellipse given by the equation

$$
\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \quad \text{where } a \geq b > 0.
$$

In this case, from [1], the non-zero coefficients of \(V\) are

$$
\begin{align*}
V_{1111} &= V_{2222} = -V_{1122} = -V_{2211} \\
&= \frac{2 \mu_0(\mu_1 - \mu_0)(a + b)^2|\omega|}{(\mu_0 + \mu_1)(a + b)^2 + (\mu_0 - \mu_1)(a - b)^2}, \\
V_{1212} &= V_{2112} = V_{1221} = V_{2121} = \frac{2 \mu_0(\mu_1 - \mu_0)(a + b)^2|\omega|}{(\mu_0 + \mu_1)(a + b)^2 + (\mu_1 - \mu_0)(a - b)^2}.
\end{align*}
$$

For the curvature moment tensor, we obtain in [3] the following result :

$$
K = 2 \lambda \mu_1(a + b)^2 + (\mu_0 - \mu_1)(a - b)^2
\times \begin{pmatrix}
\mu_1(a + b)^2 + (\mu_0 - \mu_1)(a - b)^2 & 0 \\
0 & -\chi(a, b)
\end{pmatrix},
$$

where \(\lambda\) is given by (3) and \(\chi(a, b) \geq 0\) is defined by

$$
\chi(a, b) := \int_0^{2\pi} \frac{a^2 \sin^2 \theta - b^2 \cos^2 \theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \, d\theta.
$$

In particular, simple computations yield to \(K = 0\) if \(\omega\) is a ball.

Moreover, if \(\omega \in \mathbb{R}^3\) is an ellipsoid we show that \(K\) is diagonal and, under a technical assumption, this diagonal is solution of an explicit linear system.

### 4. REFERENCES


