A MIXED FORMULATION OF THE QUASI-REVERSIBILITY METHOD

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ABSTRACT
We propose a new mixed formulation of the quasi-reversibility method to solve bidimensional and tridimensional Cauchy problems. Unlike the standard formulation of quasi-reversibility, this new formulation can be discretized using standard finite element spaces, which eases its numerical implementation. We illustrate the functionality of the method with the help of numerical experiments.

1. CAUCHY PROBLEM

For Ω a bounded open set of \(\mathbb{R}^d\) (\(d \geq 2\)), \(\Gamma \subseteq \partial \Omega\) Lipschitz, we consider the Cauchy problem: for \((f, g_d, g_n) \in L^2(\Omega) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)\), find \(u\) such that

\[
\begin{cases}
\Delta u = f & \text{in } \Omega \\
u = g_d & \text{on } \Gamma \\
\partial_n u = g_n & \text{on } \Gamma 
\end{cases}
\]

This problem is well-known to be ill-posed. It has at most one solution \(\bar{u} \in H^1(\Omega, \Delta) := \{u \in H^1(\Omega), \Delta u \in L^2(\Omega)\}\), which does not depend continuously on the data \((f, g_d, g_n)\). Hence, one needs a regularization method to solve it.

2. STANDARD FORMULATION OF QUASI-REVERSIBILITY

The quasi-reversibility method is a non-iterative regularization method of Cauchy problems, introduced by J.-L. Lions and R. Lattès in [1]. Its main idea is to approximate the ill-posed second-order Cauchy problem by a family of well-posed problems of higher order. The standard formulation of the method is based on the following variational fourth-order problem:

Problem QR: for \((f, g_0, g_1) \in L^2(\Omega) \times H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)\), for \(\varepsilon > 0\), find \(u_\varepsilon \in H^2(\Omega)\) such that

\[
\begin{cases}
u = g_d, \quad \partial_n u_\varepsilon = g_n & \text{on } \Gamma \\
(\Delta u_\varepsilon, \Delta v)_L^2(\Omega) + \varepsilon(u_\varepsilon, v)_H^2(\Omega) = (f, \Delta v)_L^2(\Omega), & \forall v \in H^2(\Omega), \\ v = \partial_n v = 0 & \text{on } \Gamma 
\end{cases}
\]

Problem QR has a unique solution \(u_\varepsilon\), and

Theorem: if the Cauchy problem admits a (unique) solution \(\bar{u}\) s.t. \(\bar{u} \in H^2(\Omega)\), then \(\lim_{\varepsilon \to 0} u_\varepsilon = \bar{u}\).

The standard quasi-reversibility method has proved effective in solving data completion and obstacles reconstruction problems [REF]. However, it has two drawbacks:

1. Theoretically, it needs a smooth exact solution (i.e. \(\bar{u} \in H^2(\Omega)\)), whereas in the most general case, \(\bar{u}\) is only in \(H^1(\Omega, \Delta)\).

2. Numerically, because it is based on a fourth-order problem, it needs \(C^1\) or nonconforming finite elements, which are rather difficult to implement and seldom available in numerical codes, especially for tridimensional problems.

To address both problems, we propose a new mixed formulation of the quasi-reversibility method.

3. MIXED FORMULATION OF QUASI-REVERSIBILITY

Let \(H_{div}(\Omega) := \{p \in L^2(\Omega)^d, \nabla p \in L^2(\Omega)\}\). The mixed formulation of quasi-reversibility is based on the following variational problem:
Problem QR\text{mixed}: for \((f, g_0, g_1) \in L^2(\Omega) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)\), for \(\varepsilon > 0\), find \((u, p) \in H^1(\Omega) \times H_{\text{div}}(\Omega)\) such that

\[
\begin{align*}
    u &= g_d, \quad p, \nu = g_n \text{ on } \Gamma \\
    (\nabla u - p, \nabla v)_{L^2(\Omega)} + \varepsilon(u, v)_{H^1(\Omega)} &= 0, \\
    \forall v \in H^1(\Omega), \quad v &= 0 \text{ on } \Gamma \\
    -(\nabla u - p, \mathbf{q})_{L^2(\Omega)} + (\nabla \cdot p, \mathbf{q})_{L^2(\Omega)} + \varepsilon(p, \mathbf{q})_{H_{\text{div}}(\Omega)} &= (f, \mathbf{q})_{L^2(\Omega)}, \\
    \forall \mathbf{q} \in H_{\text{div}}(\Omega), \quad q, \nu &= 0 \text{ on } \Gamma.
\end{align*}
\]

Problem QR\text{mixed} admits a unique solution \((u_\varepsilon, p_\varepsilon)\). Furthermore, we have the following theorem:

**Theorem**: if problem (P) admits a solution \(\bar{u}\), then

\[
    u_\varepsilon \xrightarrow{\varepsilon \to 0} \bar{u}, \quad p_\varepsilon \xrightarrow{\varepsilon \to 0} \nabla \bar{u}
\]

and we have the estimates

\[
    \|\nabla u_\varepsilon - p_\varepsilon\|_{L^2(\Omega)} \leq C(\bar{u})\sqrt{\varepsilon}, \quad \|\nabla \cdot p_\varepsilon - f\|_{L^2(\Omega)} \leq C(\bar{u})\sqrt{\varepsilon}.
\]

Note that no additional assumption on the regularity of the Cauchy problem solution is needed to obtain convergence with the mixed formulation of the quasi-reversibility. Furthermore, problem (QR\text{mixed}) is posed in \(H^1 \times H_{\text{div}}\), so it can be discretized using standard finite elements, typically Lagrange and Raviart-Thomas, which are usually available in numerical solvers. Additionally, we obtain an approximation of both the Cauchy problem solution and its gradient.

**4. NUMERICAL EXPERIMENT**

For illustration, we solve the Cauchy problem for the Laplace equation in the annulus \(\Omega := D((0,0), 1) \setminus D((0,0), 0.4)\) \((D(x, r)\) being the disc of center \(x\) and radius \(r\), with \(\Gamma = \partial D((0,0), 1)\). The searched function is \(\bar{u} = \Re\left(\frac{1}{x+iy-0.3}\right)\). We solve QR\text{mixed} problem with \(\varepsilon = 10^{-6}\), \(P_2\)-Lagrange and \(RT_1\)-Raviart-Thomas finite elements.

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**6. REFERENCES**

