INVERSE BOUNDARY PROBLEMS FOR ELLIPTIC PDE AND
BEST APPROXIMATION BY ANALYTIC FUNCTIONS

Juliette Leblond
(joint with Laurent Baratchart and Yannick Fischer)
INRIA, Sophia-Antipolis, France
Team APICS
juliette.leblond@inria.fr

ABSTRACT
Inverse boundary data transmission Cauchy type problems for two
dimensional elliptic PDE can be efficiently approached using best
constrained approximation schemes in Hardy (Hilbert) classes of anal-
lytic functions bounded on the domain’s boundary. Such recovery
results and algorithms generalize the classical planar links between
harmonic and analytic functions (of the complex variable). Specif-
ically, we will be concerned with isotropic conductivity PDE for
smooth coefficients, in annular domains. A physical application to mag-
netic plasma confinement in tokamaks will be discussed. This
is based on the works [1, 2, 3, 4, 5, 6].

1. INTRODUCTION
For smooth scalar coefficients σ, solutions to
\[
\text{div}(\sigma \text{grad} u) = \nabla \cdot (\sigma \nabla u) = 0
\]
in a disk or an annulus (or a conformally equivalent domain) of \(\mathbb{R}^2 \simeq \mathbb{C}\) coincide (up to sufficient conditions for the doubly connected case) with the real parts of “pseudo” or “generalized” analytic functions satisfying the conjugate Beltrami equation
\[
\overline{\partial} f = \nu \overline{\partial f}, \quad \nu = \frac{1 - \sigma}{1 + \sigma},
\]
with σ and ν bounded and smooth, if \(\partial, \overline{\partial}\) denote the derivative w.r.t. the complex variables \(z, \bar{z}\) [1, 4].

As in the more classical situation of harmonic functions (\(\sigma = 1\)) \(\Delta u = 0\) which are determined by the real part of analytic functions \(\overline{\partial} f = 0\) (\(\nu = 0\)), Hardy classes \(H^2_{\nu}\) of generalized analytic functions with \(L^2\) bounded traces on the boundary circles can be introduced. They correspond to \(L^2\) boundary conditions for \(u\) and its normal derivative \(\partial_n u\) (thanks to Cauchy-Riemann equations). In these Hardy classes \(H^2_{\nu}\), we establish that Dirichlet (direct) problems are well-posed as extrapolation issues from boundary data.

There, some best constrained approximation problems (bounded extremal problems) can be stated and solved, and this allows us to build robust solutions to Cauchy type issues (recovery of Cauchy boundary data or of Robin coefficients, or related free boundary issues) for the above equations [2, 3, 6]. We consider the following one: from overdetermined measurements of Dirichlet-Neumann data \(u, \partial_n u\) on a part of the boundary, recover them on the whole boundary (and in the domain). A synthesis of available results will be given.

Further, conjugate Beltrami equation serves as a model for an application to plasma shaping, in thermonuclear fusion and plasma confinement in tokamaks [5]. This example will be discussed and numerically illustrated with magnetic data from the tokamak Tore Supra, CEA-IRFM Cadarache.

2. CONCLUSIONS
Together with some perspectives, we will briefly show how:

- multiply connected situations can be handled too,
- stationary Schrödinger equations are related to conductivity and conjugate Beltrami equations,
- analytic functions can still be described and linked with harmonic ones in higher dimensions, particularly in \(\mathbb{R}^3\).
3. ACKNOWLEDGEMENTS

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4. REFERENCES


[2] L. Baratchart, J. Leblond, Y. Fischer, Dirichlet problem for $\nabla.(\sigma \nabla u) = 0$ and generalized Hardy classes in annular domains, in preparation.


