AN OBSERVER-BASED METHOD TO SOLVE THE CAUCHY PROBLEM FOR LAPLACE EQUATION

Taous Meriem Laleg-Kirati, Ammar Radwan and Chadia Zayane

King Abdulllah University of Science and technology (KAUST), Saudi Arabia
taousmeriem.laleg@kaust.edu.sa

ABSTRACT
This paper deals with the Cauchy problem for Laplace equation in a domain with a hole. The problem consists in computing the solution of Laplace equation using measurements on the outer boundary. This inverse problem is ill posed since the solution does not depend continuously on the boundary data, and small errors in the data can destroy the numerical solution. A new approach to solve this class of inverse problems is introduced. It is based on the concept of state observers, well-known in control theory.

1. INTRODUCTION
Cauchy problem for Laplace equation appears in many branches of science and engineering such as a steady state inverse heat conduction, non-destructive testing, fluid Mechanics, electric potential and electric current flow, diffusion equation and electrocardiography [2], [4], [5]. It consists in computing the solution of Laplace equation using measurements on the outer boundary. Unlike the forward problem, where the solution depends continuously on the boundary condition, the inverse problem is unstable in the sense that small changes in data can produce large or unbounded deviations in the solution [1]. Standard regularization techniques have been used to solve this inverse problem such as the Tikhonov regularization [1], [3], [9]. However, some authors choose a completely different approach. They try to reformulate the problem in one way or another to avoid, at least to some extent, the ill-posedness of the inverse problem. For example in [3], a conformal mapping that maps the studied region into an annulus where the equivalent problem is solved using a Fast Fourier Transform technique is proposed. In this case the ill-posedness is dealt with by filtering away high frequencies in the solution.

We propose in this paper to use the concept of observer, which is well-known from systems theory, to solve ill-posed inverse problems [8]. An observer enables the reconstruction of the whole system state from a few easily available measurements in real time. Many applications of observers have been proposed for example in the control field where the estimated state is used to improve the control of the system or to get an estimation of the initial state of the system.

The use of observers to solve inverse problems has been already investigated in [6] with application to inverse heat conduction problems where an observer has been designed to estimate the temperature. Recently, observers have been also considered By K. Ramdani et al [7] for recovering the initial state of an infinite dimensional system by operating only on a finite segment of output data. For this purpose, they used two observers, one working in forward time and the other working in backward time.

One advantage of using observers to solve inverse problems is that it requires solving only direct problems which are in general well-posed and well studied. Moreover, unlike optimization based methods (including regularization), observers operate recursively which implies their implementation ease and low computational cost especially when it comes to high order systems.

One has to point out that there are some conditions to satisfy before using observers. The most important is the observability of the state [7].

In the next section, we will formulate our problem and introduce our observer-based approach to solve inverse problems.

2. METHOD
In this paper, we are interested in the inverse problem for Laplace equation in a regular Lipschitz bounded domain \( \Omega \in \mathbb{R}^2 \) with a hole, where \( \Gamma_1 \) represents the outer boundary of domain, and \( \Gamma_2 \) the inner boundary. Given both Dirichlet and Neumann conditions on \( \Gamma_1 \),
In this paper, we study an inverse problem for Laplace equation using an observer-based approach. The method has been implemented for the case of an annulus domain and the first results are promising. We are also investigating the error analysis in order to find an optimal gain for the convergence of the observer.

4. REFERENCES