BEHAVIORS OF THE ENERGY OF SOLUTIONS OF THE DAMPED WAVE EQUATION

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ABSTRACT
In this talk, I will present some results on the study of the behavior of the energy of solutions of the wave equation with nonlinear damping. For a locally distributed damping and under the Geometric Control Condition of Bardos et al, we prove that the behavior of the energy is determined from a solution of an ordinary differential equation. Also we will discuss the behavior of the energy for the wave equation with arbitrary localized nonlinear damping.

1. INTRODUCTION
Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^d$ with boundary $\Gamma$. Use $A(x, \partial)$ denote a second-order strongly elliptic operator given by $A(x, \partial) = \text{div}[a(x)\partial]$ where the matrix $[a_{ij}(x)]_{i,j}$ is sufficiently smooth on $\Omega$, symmetric, and satisfies the standard uniform ellipticity condition. Consider the following wave equation with a nonlinear boundary damping:

\[
\begin{cases}
\partial_t^2 u - Au + b(t, x)g(\partial_t u) = f(t, x) & \text{in } \mathbb{R}_+ \times \Omega \\
\partial_t u = 0 & \text{in } \mathbb{R}_+ \times \partial \Omega \\
u(0), \partial_t u(0) = (u_0, u_1) & \text{on } \partial \Omega.
\end{cases}
\]

The nonlinear terms satisfy:

- $b(t, x)$ is a non negative function in $C^1(\mathbb{R}_+, L^\infty(\Omega))$.
- $g : \mathbb{R} \to \mathbb{R}$ is a continuous, monotone increasing function, $g(0) = 0$.

The force $f \in L^2_{loc}(\mathbb{R}_+, L^2(\Omega))$. We define the energy Space: $\mathcal{H} \equiv H_0^1(\Omega) \times L^2(\Omega)$. It is known that under these assumptions, for every $(u_0, u_1) \in \mathcal{H}$, the system (1) admits a unique solution in the class $u \in C(\mathbb{R}_+, H_0^1(\Omega)) \cap C^1(\mathbb{R}_+, L^2(\Omega))$.

The energy Functional is defined by

\[
E_u(t) = \frac{1}{2} \int_{\Omega} (|\nabla_A u(t, x)|^2 + |\partial_t u(t, x)|^2) \, dx.
\]

where $\nabla_A \equiv \left( \sum_{j=1}^d a_{ij}(x)\partial_x^j \right)$. The aim of this talk is to determine the behavior of this energy.

2. DAMPING IS LINEARLY BOUNDED AT INFINITY
We assume that there exist $m_0, M_0 > 0$ such that

\[
m_0 y^2 \leq g(y) y \leq M_0 y^2, \quad |y| > 1
\]

According to Lasiecka-Tataru there exists a concave strictly increasing function and linear at infinity $h_0(s)$ defined for $s \geq 0$, with $h_0(0) = 0$ and

\[
h_0(g(y)y) \geq \epsilon \left( (g(y))^2 + y^2 \right) \quad \text{for } |y| < N.
\]

Setting $m_a(\Omega_T) = \int_0^T \int_\Omega a(x) \, dx \, ds$, $K = C(T) > 0$ and

\[
h(s) = s + m_a(\Omega_T) h_0 \left( \frac{s}{m_a(\Omega_T)} \right), \quad \text{for } s \geq 0
\]

2.1. Wave equation with damping
Theorem: We assume that $b = a(x)$ and $f = 0$. Let $\omega = \{x \in \Omega : a(x) \geq \eta > 0\}$. We assume that $(\omega, T)$ satisfies the geometric control condition of Bardos et al (GCC). Let $u(t)$ is the solution to the problem (1) with initial condition $(u_0, u_1) \in H^1_0 \times L^2$. Then we have $E_u(t) \leq S(t - T)$, $t \geq T$, where $S(t)$ is the solution of

\[
\frac{dS}{dt} + \frac{1}{T} h^{-1} \left( \frac{1}{K} S \right) = 0, \quad S(0) = E_u(0)
\]
2.2. Wave equation with time dependent damping

We suppose that $b(t, x) = a(x) \rho(t)$ and $f = 0.$ Moreover we suppose that $\rho$ is monotone.

**Theorem:** We assume that $(\omega, T)$ satisfies the assumption (GCC).

Let $u(t)$ be the solution to the problem \([\square]\) with initial condition $(u_0, u_1) \in H_0^1 \times L^2.$ Then we have $E_u(t) \leq S(t - T), \ t \geq T,$ where $S(t)$ is the solution of

$$
\frac{dS}{dt} + \beta(t) h^{-1} \left( \frac{\alpha(t)}{K} S(t) \right) = 0, \quad S(0) = E_u(0),
$$

where

$$
\beta(t) = \begin{cases} \frac{1}{2} \rho(t + T) & \rho \nearrow \alpha(t), \\
\frac{1}{2} \rho(t - T) & \rho \searrow \end{cases}
$$

Moreover, if for some $T_0 > 1$ and $\forall 0 < \gamma << 1,$

$$
\int_{T_0}^t \beta(t) h^{-1} \left( \frac{\alpha(t)}{K} \right) ds \rightarrow +\infty \quad \text{Then} \quad E_u(t) \rightarrow 0
$$

2.3. Wave equation with damping and external force

**Theorem:** We assume that $b(t, x) = a(x)$ and $(\omega, T)$ satisfies the assumption (GCC) and

$$
\Gamma(t) = \|f(t, \cdot)\|_{L^2(\Omega)}^2 + \psi^* \left( \|f(t, \cdot)\|_{L^2(\Omega)} \right) \in L_{loc}^1(\mathbb{R}^+)
$$

where $\psi^*$ is the convex conjugate of the function $\psi,$ defined by

$$
\psi(s) = \begin{cases} \frac{1}{2} h^{-1} \left( \frac{s^2}{80 \alpha \epsilon^4} \right) & s \in \mathbb{R}_+, \\
\infty & s \in \mathbb{R}^-
\end{cases}
$$

Let $u(t)$ be the solution to the non-linear problem \([\square]\) with initial condition $(u_0, u_1) \in H_0^1 \times L^2.$ Then we have

$$
E_u(t) \leq 2e^T \left( S(t - T) + \int_{t-T}^t \Gamma(s) ds \right), \quad t \geq T
$$

where $S(t)$ is the solution of

$$
\frac{dS}{dt} + \frac{1}{4T} h^{-1} \left( \frac{S(t)}{K} \right) = \Gamma(t), \quad S(0) = E_u(0).
$$

2.4. Wave equation with arbitrary localized damping

**Theorem:** $u(t)$ is the solution to the non-linear problem \([\square]\) with initial condition $(u_0, u_1) \in X = (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega).$ Then we have $E_u(t) \leq S(t - T), \ t \geq T,$ where $S(t)$ is the solution of

$$
\frac{dS}{dt} + \frac{1}{T} h^{-1} \psi^{-1} \left( \frac{S(K)}{K} \right) = 0, \quad S(0) = E_u(0).
$$

where $K = C(T, \|(u_0, u_1)\|_X)$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}_+,$ strictly increasing function, defined by

$$
\psi(s) = \left( \ln \left( \frac{1}{s} + 1 \right) \right)^{-2} + s; \ 0 < \beta < 1.
$$

3. SOME REMARKS

We assume that $f = 0$ and $b = a(x).$

3.1. Damping is not linearly bounded

We have to find $p > 2,$ such that $\partial_t u \in L^\infty(\mathbb{R}_+, L^p(\Omega)),$ so regular initial data. In this case the function $h(s) = h_1(s) + m_u(\Omega_T) h_0 \left( \frac{s}{m_u(\Omega_T)} \right),$ where $h_1$ depends on the parameter $p$ and the behavior of the damping. Noting that if $p = \infty,$ we have $h_1(s) = s.$

3.2. Global distributed damping

When the damping is globally distributed:

- If $g$ is linearly bounded or sublinear at infinity, we obtain the same rate of decay as in the case of a localized damping (Under the GCC).
- If $g$ is superlinear we have to find $p > 2,$ such that $u \in L^\infty(\mathbb{R}_+, L^p(\Omega)).$

4. REFERENCES