Identification of a time-dependent point source in a linear transport equation with spatially varying coefficients: detection of a pollution source

Adel Hamdi and Imed Mahfoudhi
Laboratoire LMI EA 3226 - INSA de Rouen France

Motivation: Environmental monitoring

Mathematical modelling:

The BOD concentration, denoted here by $w$, is governed by the following system, see [1,2,3]:

\[ L[u](x,t) = \lambda D(x,u) + \text{Noise intensity} \quad \text{in } (0,t) \times (0,T) \]

\[ u(x,0) = 0 \quad \text{for } 0 < x < \ell \]

\[ u(0,t) = \partial_u u_0(t) = 0 \quad \text{for } 0 < t < T \]

$\lambda$ is the Dirac mass and $L$ is the parabolic differential operator

\[ L[u](x,t) = \partial_t(u(x,t) - \delta(x,t)u(x,t)) + \rho(x,t)u(x,t) \]

Here, the reaction coefficient $\rho$ is a positive real number, the spatially varying velocity $v$ is such that $v \in C^2(0,T)$ and the spatially varying diffusion $D$ is a positive and piecewise $C^2$-class function of $[0,\ell]$. Furthermore, $S$ represents the source location whereas $\lambda \in C^2(0,T)$ is its time-dependent intensity function.

Problem statement:

Nonlinear inverse source problem:

Assuming available some observations of the state $u$, we aim to localize the position $S$ of a sought source and recover the historic with respect to the time of its intensity function $\lambda$.

Due to the linearity of the operator $L$ and in view of the superposition principle, the use of a non-zero initial condition and/or inhomogeneous boundary conditions do not affect the results established in this study.

Identification:

Theorem:

Under the conditions of the previous theorem and given the records $[d_0(t), d_1(t), \ldots, d_T(t)]$, for $0 \leq t \leq T$ of the state $u$ at the two observation points $a$ and $b$, the setting $\Phi(S, \lambda) = \{d_0(t), d_1(t), \ldots, d_T(t)\}$ for $0 \leq t \leq T$ implies that the source elements $S$ and $\lambda$ are subject to

\[ \Phi(S) \sim Q_2 \quad \text{and} \quad \int_0^T \lambda \partial t dt = \frac{Q_2}{w(S)(\lambda)} \]

Identification procedure:

- The assertion (1) implies that $\Phi$ involved in (2) is a continuous and strictly monotonic function. Therefore, the sought solution $S$ is determined from (2) using Newton's method.
- Then, using the already computed position $S$ we transform the task of identifying $\lambda$ into solving a deconvolution problem.

Numerical Experiments:

Experiments with $v = 0.1\text{ms}^{-1}$, $r = 2 \times 10^{-3}\text{ms}^{-1}$, $D(x) = d_m + \alpha(1 - e^{-\beta x})$, $d_m = 10^{-5}\text{m}^2\text{ms}^{-1}$, $\alpha = 10^3$, $\beta = 10^{-2}$ and $\mathcal{S} = 64$.

References: