Plasma boundary reconstruction using
topological asymptotic expansion

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Abstract
The Tokamak is an experimental machine which aims to confine the plasma in a magnetic field to control the nuclear fusion of atoms of mass five. The real-time reconstruction of the plasma magnetic equilibrium in a Tokamak is a key step to access high-performance regimes.

Tokamak
In an asymmetric configuration, the plasma equilibrium is described by the equation (see [3])
\[ L_0 v = 0 \quad \text{in } \Omega, \]
where \( \Omega = \Omega_0^{1,2} \) is the vacuum region surrounding the plasma \( \Omega \), \( \Omega \) is the vacuum vessel, and \( L \) is the Grad-Shafranov operator
\[ L = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial v}{\partial \rho} \right) + \frac{\partial v}{\partial z}. \]
Due to its economic importance, the plasma control problem has long been receiving considerable attention by engineers and mathematicians [4, 8]. Therefore, the most developed methods deal with theory control or parameter optimization.

In this work, we propose a new method. Our approach is based on the topological sensitivity analysis [1, 2, 5, 6, 7]. The plasma domain defined by a level curve of a scalar function, called the topological gradient. The topological gradient is calculated from a topological asymptotic expansion for the Grad-Shafranov operator. The proposed approach leads to a fast and accurate numerical algorithm. The efficiency of the proposed method is illustrated by some numerical examples.

I The inverse problem
We consider here the inverse problem of determining plasma boundary \( \Gamma_p \) location from over-specified boundary data on \( \Gamma = \partial \Omega_0 \). Knowing a complete set of Cauchy data, the poloidal flux \( \psi \) satisfies the system
\[ \begin{align*}
L_0 v &= 0 \quad \text{in } \Omega, \\
\rho v &= \psi \quad \text{on } \Gamma, \\
\rho v &= 0 \quad \text{on } \Gamma_p
\end{align*} \]
and \( \Gamma = \partial \Omega_0 \). The considered inverse problem can be formulated as follows: given boundary data \( \psi^0, \psi^1 \), find the optimal location of the plasma boundary \( \Gamma_p \).

II Topological sensitivity analysis
The topological sensitivity analysis method consists in studying the variation of a cost function \( j(\Omega) \) with respect to the insertion of small holes \( \omega_\varepsilon = \omega + \varepsilon \mu_\omega \) in the domain \( \Omega \), where \( \omega = \omega(X, y) \subset \mathbb{R}^2 \). The level set of \( \omega \subset \mathbb{R}^2 \) is a fixed bounded domain containing the origin, whose boundary is connected and piecewise of class \( C^2 \).

The topological gradient method leads to an asymptotic expansion of the form
\[ j(\Omega) \sim j(\Omega^0) + \sum_{n=1}^{\infty} j^{(n)}(\Omega^0) \varepsilon^n, \]
where \( j^{(n)}(\Omega^0) \) is a scalar function with respect to the parameter \( \varepsilon \).

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Theorem: The function \( j \) admits the following asymptotic expansion
\[ j(\Omega) = j(\Omega^0) + \frac{1}{\varepsilon} j^1(\Omega^0, \varepsilon) + o\left(\frac{1}{\varepsilon}\right) \]
where \( j^1(\Omega^0, \varepsilon) \) is a function of the small parameter \( \varepsilon \).

IV Conclusion
In this work, we prove a fast and efficient identification procedure. Our numerical algorithm is based on the asymptotic expansion established in the previous Theorem. The unknown plasma boundary is defined by level set curve of the topological gradient \( g(X) \) defined for all \( X = (x, y) \in \Omega \). Our identification procedure is a one-shot algorithm based on the following steps:

- solve the direct and adjoint problems;
- compute the topological gradient \( g \);
- determine the plasma boundary \( \Gamma_p = \{x \in \Omega, g(x) = (1 - r)g_{\max} < 0\} \)

where \( g_{\max} = \max g(X) \) and \( r \in (0, 1] \). It is chosen in such a way that the function \( r \) decreases as much as possible.

First example
In this case, we used the following data:
- The vacuum vessel region is defined by the disc \( \Omega = B(\Omega_0, 1) \), with \( C = (2,0) \).
- The exact plasma domain is defined by the disc \( D^0 = B(C, 0.2) \), with \( C = (1.5, -0.1) \).
- The Dirichlet and Neumann boundary data are given by:
\[ \begin{align*}
\psi_\Omega(x, y) &= \sin(0.5 \pi \sqrt{x^2 + y^2}), \\
\psi_{\partial \Omega}(x, y) &= \psi(x, y), \\
\psi_{\partial \Gamma_p}(x, y) &= \min(g(x), 0) + \mu_{\Omega} \mu_{\Gamma_p}
\end{align*} \]

To evaluate the accuracy of our approach, we introduce the following error function which defines the Hausdorff distance between the exact \( \Gamma_p \) and obtained \( \Gamma_p \) plasma domain
\[ e(r) = \max(D(\Omega, 0.2)) - \max(D(\Omega, r) \cap D(\Omega, 0.2)), \]
where \( D(\Omega, r) = \{x \in \Omega, g(x) < (1 - r)g_{\max}\} \).

The variation of the error function \( e(r) \) is illustrated in Figure. As we can see in Figure, the optimal choice of the parameter \( r \) is \( r = 0.98 \).

Second example
In this case, the unknown plasma is defined by the ellipse \( \{x \in \mathbb{R}^2 \mid \rho^2(x) + \eta^2(r) = 1, \rho > 0, \eta > 0\} \) with \( C = (2,0) \), \( \rho_1 = 0.4 \), \( \eta_1 = 0.5 \).

One: Isoclines of \( \psi \). Two: The topological gradient \( g \) and those: Variance of the error function \( e(r) \).

References