1. Introduction

A Mumford-Shah like Approach for Finding the Support and Intensity of a Photon Source Inside an Organism

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2. Analysis of the Minimization Problem

Existence of a Minimizer:

Theorem 2.1. For all \( \alpha > 0 \) and \( \lambda \in L^2(\Omega) \) there exists a solution \((\lambda^\bullet, G^\bullet) \in \Lambda \times L\) of the problem (2), i.e.,

\[ f_\alpha(\lambda, G) \leq f_\alpha(\lambda^\bullet, G^\bullet) \quad \text{for all } (\lambda, G) \in \Lambda \times L. \]

Stability:

Theorem 2.2. Let \( \{\lambda_n\} \rightarrow \lambda^\bullet \) in \( L^2 \) as \( n \to \infty \) and \((\lambda_n, G_n)\) minimize

\[ f_\alpha(\lambda, G) \; : \; \frac{1}{2} k^2(\lambda, G) + \alpha \text{Per}(G) \quad \text{over } \Lambda \times L. \]

Then there exists a subsequence \((\lambda^\bullet, G^\bullet)\) converging to a minimizer \((\lambda^\bullet, G^\bullet)\) of \( f_\alpha \) in the sense that

\[ k^{\lambda_n} \chi_{\Omega_n} - \lambda^\bullet \chi_{\Omega} \rightarrow 0 \quad \text{as } n \to \infty. \]

Furthermore, every convergent subsequence of \((\lambda_n, G_n)\) converges to a minimizer of \( f_\alpha \).

Regularization Property:

Theorem 2.3. Let \( \lambda \) be in the range of \( F \) and choose the regularization parameter according to \( \delta \sim \alpha / \lambda^2 \) where

\[ k^{\lambda^\bullet} \chi_{\Omega^\bullet} \rightarrow 0 \quad \text{and} \quad \text{dist}(\lambda, \Lambda) \rightarrow 0 \quad \text{as } \alpha \to 0. \]

In addition, let \( \{\lambda_n\} \) be a positive null sequence and \( \{\Delta_n\} \) such that

\[ k^{\lambda_n} \ominus g^{\lambda_n} \leq \Delta_n. \]

Then, with the notation of Theorem 2.2, the sequence \( \{\lambda_n, G_n\} \) of minimizers of \( f_{\alpha_n} \) possesses a subsequence converging to \((\lambda^\bullet, G^\bullet)\) which satisfies

\[ G^\bullet = \arg \min \{\text{Per}(G) \mid G \in L^2\}, \]

where

\[ \lambda^\bullet \in \{\lambda \in L^2 \mid F(\lambda, G^\bullet) = 0\}. \]

Furthermore, every convergent subsequence of \((\lambda_n, G_n)\) converges to a \( (\lambda^\bullet, G^\bullet) \) with property (3).

3. Existence of Smooth Almost Stationary Points

Approximate Variational Principle:

Theorem 3.1. Let \((\lambda^\bullet, G^\bullet)\) be a minimizer of \( f_\alpha \) and \( \lambda^\bullet \) an inner point of \( \Lambda \). In the three-dimensional case, assume that \( G^\bullet \) is a finite union of disjoint connected domains. Then for any \( \varepsilon > 0 \) sufficiently small we can find an intensity \( \lambda^\bullet \in \Lambda \) and a \( C^2 \)-domain \( G^\bullet \) satisfying

\[ f_\alpha(\lambda^\bullet, G^\bullet) - f_\alpha(\lambda^\bullet, G^\bullet) \leq \varepsilon, \quad k^{\lambda^\bullet} \chi_{\Omega^\bullet} - \lambda^\bullet \chi_{\Omega} \leq \varepsilon, \quad k^{\lambda^\bullet} \chi_{\Omega^\bullet} G^\bullet \chi_{\Omega^\bullet} \leq \varepsilon. \]

Herein, \( \delta_G f_\alpha(\lambda^\bullet) \) denotes the domain derivative of \( f_\alpha(\lambda^\bullet) \).

4. Numerical Experiments

Restriction to Star-shaped Domains: For the numerical experiments we only consider star-shaped domains and work on the linear space of parametrizations. All previous results hold true in this setting.

Implementation: The discussed approach is implemented using trigonometric polynomials as parametrization of the domain and a projected gradient method for the minimization.

References


Figure 1: Bioluminescence Image. Kindly provided by the group of Prof. Sahin at University Medical Center, Mainz.

Figure 2: Reconstruction (blue solid) and original source (red dashed) with \( \alpha = 0.0075 \) after 37 gradient iterations.

Figure 3: Reconstruction (blue solid) and original source (red dashed) with \( \alpha = 0.0075 \) after 436 gradient iterations assuming a different midpoint.

Figure 4: Reconstruction (blue solid) and original source (red dashed) with \( \alpha = 0.0075 \) after 436 gradient iterations assuming a different midpoint.