Optimal Control Problem

Introduction: Let $f: \mathbb{R} \to \mathbb{R}$ be a $C^1$ strictly convex function having super linear growth. That is

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty.$$  

Let $u_1, u_0 \in L^2(\mathbb{R})$ and consider the scalar conservation law

$$u_t + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x, 0) = u_0(x).$$  

(2)

In general, (2) does not admit classical solutions and hence look for weak solutions. This problem was well studied and showed that (2) admits a unique weak solution satisfying Lax-Oleinik-Kruzkov entropy conditions. In the sequel we mean $u$ a solution of (2) if it is a weak solution satisfying entropy condition. In [7] following optimal control problem associated to (2) had been considered.

Let $u \in L^2(\mathbb{R}^+)$ be a target function and $A \subset L^2(\mathbb{R})$, a set of admissible controls. Let $u_0$ be the associated solution of (2) and $T > 0$. Define the cost functional $J$ on $A$ by

$$J(u) = \int_0^T |u_t(x) - h(x)|^2 \, dx.$$  

(3)

Then the optimal control problem is to find an $u \in A$ such that

$$J(u) = \inf_{u \in A} J(u).$$  

(4)

Under a suitable conditions on $k$ and $A$, they prove that $u_0$ exist. In general $u_0$ is not unique. The basic problem related to this is to capture a minimizer. $u_0$ is a very hard problem because the cost functional $J$ is highly nonlinear, non differentiable and non convex. For the Burger's equation in [7], they have proposed a numerical scheme called “alternating descent direction” by using the linearization technique developed in [8][9]. In that work, convergence analysis is completely open. Here we have tackled this problem in a completely different way. We have modified our cost functional

$$J(u) = \int_0^T f(u_t(x), T) - f(h(x)) \, dx.$$  

(5)

Optimal control problem : Given $A$ as above, find $u \in A$ such that

$$J(u) = \min_{u \in A} J(u)$$  

(6)

and if the minimizer exist, then device a scheme to capture it. Then we have the following main result.

Main Theorem : There exists a minimizer for (6) which can be captured by using the standard convex optimization problem in a Hilbert space and backward algorithm.

Techniques involves in the proof :

1) Due to our modified cost function $J$, the optimal control problem reduces to the standard convex optimization problem via Lax-Oleinik explicit formula. By Hilbert projection Theorem minimizer exists.

2) Then we use the backward construction introduced in [2] to obtain an optimal solution.

The novelty of this method is that it is constructive and easy to derive a numerical scheme to capture an optimal solution.

Exact Control Problem

Introduction : We consider the following scalar conservation law in one space dimension,

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty.$$  

(7)

Let $T > 0$, $0 \leq \xi < T$, $\xi < A < B$, $I = (A, B)$, $\Omega = I \times (0, T)$, $u_1, u_2 \in L^2(I)$, $b_0, b_1 \in L^2([0, T])$ and consider

$$u_0 + f(u)_t = 0, \quad x \in I,$$
$$u(x, T) = u_1(x), \quad u(x, 0) = u_0(x).$$  

(8)

(9)

(10)

(11)

The problem for exact controllability of initial and initial boundary value problem was open for quite a long time. Normally for the non linear evolution equations, technique of linearization is adopted to study controllability problems. Unfortunately this method does not work (see Horisn [9]) and very few results are available on this subject. Here we consider the following three problems of controllability. Let $u_0 \in L^2(\mathbb{R})$ then

1) Controllability for pure initial value problem : Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\int_{\mathbb{R}} f(x) \, dx = 0,$$
$$u(x, T) = u_1(x), \quad x \in I.$$  

(12)

$$u(x, 0) = \begin{cases} u_0(x) & \text{if } x \in J_1, \\ 0 & \text{if } x \notin J_2. \end{cases}$$  

(13)

Similarly we can define

2) Controllability for one sided initial boundary value problem:

3) Controllability in a strip :

Now the question is whether the problems (1), (2) and (3) admit a solution? In fact, it is true and we have settled all the three problems.

In the case of problem (2), Ancona et al. [3][4] studied the problem from the point of view of Hamilton-Jacobs equations and studied the compactness properties of $u(x, T)$ when $u(x, 0)$ is $u_0$ and $u(x, T)$ is $u_1$. Here $u$ is a set of controls satisfying some properties. First we define the Target space for initial value problem (9A):

$$L(A) = \{ \phi \in L^2(I) : \int \phi(x) \, dx = \rho \}$$

(15)

Main Theorem : Let $A_1 = (C_1, C_2)$, $A_2 = (B_1, B_2)$, $\rho \in L^2(I)$, $\rho$ is a non-decreasing function.

$$A_1 \leq \rho \leq A_2, \quad \forall \rho \in A_1.$$  

(16)

Then there exists an $u_0 \in L^2(I)$ such that $(u, u_0)$ is a solution to problem (7).

Techniques involves in the proof :

1) Superlinearity of $J$ plays an important role in removing the condition on $T$ and by creating free regions (free region Lemma).

2) Next using convexity and backward construction, we explicitly construct solutions in these free regions for particular data which allow to obtain solutions for control problems.

References


