Problem

Let Ω be an open and bounded subset of \( \mathbb{R}^d \) with smooth boundary \( \Gamma \).
We are interested in the stabilization properties of systems of hyperbolic-type equations, such as
\[
\begin{align*}
\begin{cases}
  u_{tt} - \Delta u + \beta u + \alpha v + \omega v &= 0 & \text{in } \Omega \times \mathbb{R} \\
  v_{tt} - \Delta v + \alpha u &= 0 & \text{in } \Omega \times \mathbb{R}
\end{cases}
\end{align*}
\]
with hybrid boundary conditions on \( \Gamma = \partial \Omega \), for example \( \frac{\partial u}{\partial n} = 0 = v \) on \( \Gamma \times \mathbb{R} \), where \( \beta > 0 \) and \( \alpha \in \mathbb{R} \).

Indirect Stabilization

When dealing with systems involving quantities described by several components, pretending to control or observe all the state variables might be unrealistic. In applications to mathematical models for the vibrations of flexible structures, electromagnetism, or fluid control, it may happen that only part of such components can be observed. This is why it becomes essential to study whether controlling only a reduced number of state variables suffices to ensure the stability of the full system. Indeed, in the case of system (1), for \( \alpha = 0 \), only the first component \( u \) is damped, while the second component \( v \) is conservative.

Lack of exponential stability

The total energy associated with system (5) is
\[
E(t) := E_1(u) + E_2(v, q) + \alpha(u, v)
\]
for all \( U \in H \), where
\[
E_1(u, p) = \frac{1}{2} \left( |A_1 u|^2 + |p|^2 \right)
\]
for all \((u, p) \in (A_1^{-1} H) \times H (i = 1, 2)\).

In [1] the authors proved that exponential stability fails for system (5), or, equivalently, for system (2), owing to the coupling of the damping operator in the energy space.

Abstract setting for evolution equations in Hilbert space

Let \((H, \langle \cdot, \cdot \rangle)\) be a real separable Hilbert space, with norm \( ||\cdot|| \). Consider on \( H \) the system of evolution equations
\[
\begin{align*}
\begin{cases}
  u''(t) + A_1 u(t) + B u'(t) + \alpha v(t) = 0 & t \in \mathbb{R} \\
  v''(t) + A_2 v(t) + \alpha u(t) = 0 & t \in \mathbb{R}
\end{cases}
\end{align*}
\]
where

(H1) \( A_i : D(A_i) \subset H \to H (i = 1, 2) \) are densely defined closed linear operators such that
\[
A_i = A_i^* , \quad \langle A_i u, u \rangle \geq \omega_i ||u||^2 \quad \forall u \in D(A_i) \quad \text{for some } \omega_1, \omega_2 > 0.\]

(H2) \( B \) is a bounded linear operator on \( H \) such that
\[
B = B^* , \quad \langle B u, u \rangle \geq \beta ||u||^2 \quad \forall u \in H \quad \text{for some } \beta > 0.\]

(H3) \( \alpha \) is a real number such that \( 0 < ||\alpha|| < \sqrt{\omega_1 \omega_2} \).

System (2), with initial conditions
\[
\begin{align*}
\begin{cases}
  u(0) = u^0 \in D(A_1^{1/2}) , & u'(0) = u^1 \in H , & v(0) = v^0 \in D(A_2^{1/2}) , & v'(0) = v^1 \in H .
\end{cases}
\end{align*}
\]
can be formulated as a first order Cauchy problem on \( H := (A_1^{1/2} H) \times (A_2^{1/2} H) \times H \), defining \( D(L) := (A_1 \times (A_2^{1/2}) \times (A_2^{1/2} \times A_2^{1/2}) \), and \( A : D(A) \subset H \to H \) by
\[
AU = (p, -A_2 u - Bp - \alpha v, q, -A_2 v - \alpha u) \quad \forall U = (u, p, v, q) \in D(A).
\]
Then, problem (2) with initial conditions (3) takes the equivalent form
\[
U'(t) = AU(t) , \quad U(0) = U_0 := (u^0, u^1, v^0, v^1).
\]

Main Result

Theorem. Under the compatibility condition \( D(A_2) \to D(A_1^{1/2}) \),

i) if \( U_0 \in D(A_0), \quad n \geq 1 \), then \( \varepsilon(t) \leq C_{\varepsilon} n^{-\frac{n}{2}} \sum_{k=0}^{n} \varepsilon(t_k) (0) \quad \forall t > 0.\)

ii) if \( U_0 \in (H, D(A_0))_{n, 2}, \quad n \geq 1, \quad 0 < \theta < 1 \) then \( \varepsilon(t) \leq C_{\varepsilon} \frac{n}{||U_0||_{H, D(A_0)}^2} \quad \forall t > 0.\)

The proof of this Theorem is based on some key topics:

- the dissipation relation for the total energy \( \varepsilon(U) \);
- multipliers of the form \( A_1^{-1} v \) and \( A_2^{-1} u \);
- an abstract lemma on an integral inequality (see [1]);
- interpolation techniques.

Applications

Thanks to the previous Theorem we can ensure polynomial decay rate for the energy of many systems of PDEs with hybrid boundary conditions, such as

1. \( A_1 u = -\Delta u + \lambda u \), where \( \lambda > 0 \), \( A_2 u = -\Delta v \), with \( \frac{\partial u}{\partial n} = 0 = v \) on \( \Gamma \times \mathbb{R} \).

   In this case, we have
   \[
   D(A_1) = \left\{ u \in H^{2}(\Omega) : \frac{\partial u}{\partial n} = 0 \right\} , \quad D(A_2) = H^2(\Omega) \cap H_0^1(\Omega) ;
   \]

2. \( A_1 u = \Delta^2 u + \lambda u \), where \( \lambda > 0 \), \( A_2 u = -\Delta v \), with \( \frac{\partial u}{\partial n} + \frac{\partial v}{\partial n} = 0 = v \) on \( \Gamma \times \mathbb{R} ;
   \]

3. \( A_1 u = -\Delta u, A_2 u = -\Delta v, \) with \( \frac{\partial u}{\partial n} + \frac{\partial v}{\partial n} = 0 = v \) on \( \Gamma \times \mathbb{R} ;
   \]

4. \( A_1 u = -\Delta u, A_2 u = \Delta^2 v, \) with \( \frac{\partial u}{\partial n} + \frac{\partial v}{\partial n} = 0 = v \) on \( \Gamma \times \mathbb{R} ;
   \]

5. \( A_1 u = -\Delta u, A_2 u = \Delta^2 v, \) with \( \frac{\partial u}{\partial n} + \frac{\partial v}{\partial n} = 0 = v \) on \( \Gamma \times \mathbb{R} ;
   \]

6. \( \Gamma_0 \subset \Gamma \subset \Omega \) such that \( \Gamma_0 \cap \Gamma_1 = \emptyset \), \( A_1 u = -\Delta u, A_2 u = -\Delta v \), with boundary conditions \u = 0 \) on \( \Gamma_0 \times \mathbb{R} \), \( \frac{\partial u}{\partial n} = 0 = \Gamma_1 \times \mathbb{R} , v = 0 \) on \( \Gamma \times \mathbb{R} ;
   \]

Remark. Examples of PDEs with the same kind of boundary conditions on \( u \) and \( v \) were already covered by the compatibility condition \( D(A_2) \to D(A_1^{1/2}) \) for some integer \( j \geq 2 \) introduced in [1].

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