Transmission Eigenvalues in Inverse Scattering Theory

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Scattering by an Inhomogeneous Media

\[
\begin{align*}
\Delta u^s + k^2 u^s &= 0 & \text{in } \mathbb{R}^m \setminus \overline{D} \\
\nabla \cdot A \nabla u + k^2 nu &= 0 & \text{in } D \\
u &= u^s + u^i & \text{in } \partial D \\
\nu \cdot A \nabla u &= \nu \cdot \nabla (u^s + u^i) & \text{in } \partial D \\
\lim_{r \to \infty} r^{m-1} \left( \frac{\partial u^s}{\partial r} - iku^s \right) &= 0
\end{align*}
\]

\[A, n\] represent the inhomogeneous media, here meant in a general sense.

**Question:** Is there an incident wave \( u^i \) that does not scatter?

The answer to this question leads to the transmission eigenvalue problem.
If there exists a nontrivial solution to the homogeneous interior transmission problem

\[
\Delta v + k^2 v = 0 \quad \text{in} \quad D
\]
\[
\nabla \cdot A \nabla w + k^2 n w = 0 \quad \text{in} \quad D
\]
\[
w = v \quad \text{on} \quad \partial D
\]
\[
\nu \cdot A \nabla w = \nu \cdot \nabla v \quad \text{on} \quad \partial D
\]

such that \( v \) can be extended outside \( D \) as a solution to the Helmholtz equation \( \tilde{v} \), then the scattered field due to \( \tilde{v} \) as incident wave is identically zero.

Values of \( k \) for which this problem has non trivial solution are referred to as transmission eigenvalues.
Transmission Eigenvalues

In general such an extension of $\nu$ does not exits!

Since superposition of plane waves so-called Herglotz wave functions

$$\nu_g(x) := \int_{\Omega} e^{ikx \cdot d} g(d) ds(d), \quad \Omega := \{d : |d| = 1\}$$

or superposition of point sources

$$\mathcal{S}_\varphi(x) := \int_\Lambda \varphi(y) \Phi(x, y) ds_y, \quad \Lambda \text{ is a surface in } \mathbb{R}^m \setminus \overline{D}$$

where $\Phi(x, y)$ is the fundamental solution of the Helmholtz equation,

are dense in $\{\nu \in \mathcal{H}(D) : \Delta \nu + k^2 \nu = 0 \text{ in } D\}$,

at a transmission eigenvalue there is an incident field that produces arbitrarily small scattered field

(here $\mathcal{H}(D)$ is either $L^2(D)$ or $H^1(D)$).
Motivation

Two important issues:

- Real transmission eigenvalues can be determined from the scattering data.
- Transmission eigenvalues carry information about material properties.

Therefore, transmission eigenvalues can be used to quantify the presence of abnormalities inside homogeneous media and use this information to test the integrity of materials.

How are real transmission eigenvalues seen in the scattering data?
Measurements

Since transmission eigenvalues correspond to "non scattering" frequencies, at a transmission eigenvalue the (modified) measurement operator fails to be injective. Exploring this, gives a way to see transmission eigenvalues in the scattering data.

To fix our ideas consider

- the far field operator $F : L^2(\Omega) \to L^2(\Omega)$

$$(Fg)(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d, k) g(d) ds(d), \quad \left( S = I + \frac{ik}{\sqrt{2\pi k}} e^{-i\pi/4} F \right)$$

where $u_\infty(\hat{x}, d, k)$ is the far field of the scattered field $u^s(x, d, k)$ due to a plane wave $u^i(x) = e^{ikx \cdot d}$, for $\hat{x}, d \in \Omega$, and $k \in [k_0, k_1]$

- and the far field equation

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z, k), \quad g \in L^2(\Omega)$$
Computation of Real TE

Assume that \( z \in D \) and \( \delta > 0 \) is the measurement noise level.

Let \( g_{z,\delta,k} \) be the Tikhonov regularized solution of the far field equation, i.e the unique minimizer

\[
\|F_\delta g - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)}^2 + \epsilon(\delta)\|g\|_{L^2(\Omega)}^2, \quad \epsilon(\delta) \to 0 \text{ as } \delta \to 0
\]

- If \( k \) is not a transmission eigenvalue then

\[
\lim_{\delta \to 0} \|v_{g_{z,\delta,k}}\|_{\mathcal{H}(D)} \text{ exists.}
\]


- If \( k \) is a transmission eigenvalue then

\[
\lim_{\delta \to 0} \|v_{g_{z,\delta,k}}\|_{\mathcal{H}(D)} = \infty.
\]

Cakoni-Colton-Haddar, Comp. Rend. Math. 2010
Computation of Real TE

A composite plot of $\|g_{z_i}\|_{L^2(\Omega)}$ against $k$ for 25 random points $z_i \in D$

The average of $\|g_{z_i}\|_{L^2(\Omega)}$ over all choices of $z_i \in D$.

Computation of the transmission eigenvalues from the far field equation for the unit square $D$. 
Historical Overview

- The transmission eigenvalue problem in scattering theory was introduced by Kirsch (1986) and Colton-Monk (1988).

- Research was focused on the discreteness of transmission eigenvalues for a variety of scattering problems:
  
  Colton-Kirsch-Päivärinta (1989) – many more.....

  In the above work, it is always assumed that either $n - 1 > 0$ or $1 - n > 0$ in $D$ (may be zero at the boundary $\partial D$).

- The first proof of existence of at least one transmission eigenvalue for large enough contrast is due to Päivärinta-Sylvester (2009).

- The existence of an infinite set of transmission eigenvalues is proven by Cakoni-Gintides-Haddar (2010) under only assumption that either $n - 1 > 0$ or $1 - n > 0$. The existence has been extended to other scattering problems by Kirsch (2009), Cakoni-Haddar (2010) Cakoni-Kirsch (2010), Bellis-Cakoni-Guzina (2011), Cossonniere (Ph.D. thesis) etc.
Cakoni-Colton-Haddar (2010) and Cossonriere-Haddar (2011) have studied the case when inside media there are subregions with the same material properties as the background.

Hitrik-Krupchyk-Ola-Päivärinta (2010), in a series of papers have extended the transmission eigenvalue problem to a more general class of differential operators with constant coefficients.

Finch has connected the discreteness of the transmission spectrum to a uniqueness question in thermo-acoustic imaging for which \( n - 1 \) can change sign.

Sylvester (2012) has shown that the set of transmission eigenvalues is at most discrete if \( A = I \) and \( n - 1 \) is positive (or negative) only in a neighborhood of \( \partial D \) but otherwise could change sign inside \( D \). A similar result is obtained by Bonnet Ben Dhia - Chesnel - Haddar (2011) using T-coercivity and Lakshtanov-Vainberg (to appear), for the case \( A - I \) and \( n - 1 \) keep fixed sign in a neighborhood at the boundary \( \partial D \).
Recall the transmission eigenvalue problem (set $A = I$)

\[
\Delta w + k^2 n w = 0 \quad \text{in} \quad D \\
\Delta v + k^2 v = 0 \quad \text{in} \quad D \\
w = v \quad \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D
\]

It is a nonstandard eigenvalue problem

\[
\int_D (\nabla w \cdot \nabla \overline{\psi} - k^2 n(x) w \overline{\psi}) \, dx = \int_D (\nabla v \cdot \nabla \overline{\phi} - k^2 v \overline{\phi}) \, dx
\]

- If $n = 1$ the interior transmission problem is degenerate
- If $\Im(n) > 0$ in $\overline{D}$, there are no real transmission eigenvalues.
Let $u = w - v$, we have that

$$\Delta u + k^2 nu = k^2 (n-1)v.$$

Applying $(\Delta + k^2)$, the transmission eigenvalue problem can be written for $u \in H^2_0(D)$ as an eigenvalue problem for the fourth order equation:

$$(\Delta + k^2) \frac{1}{n-1} (\Delta + k^2 n) u = 0$$

i.e. in the variational form

$$\int_D \frac{1}{n-1} (\Delta u + k^2 nu)(\Delta \varphi + k^2 \varphi) \, dx = 0 \quad \text{for all } \varphi \in H^2_0(D)$$
Transmission Eigenvalues

Assuming $n$ real valued and letting $k^2 := \tau$, the variational formulation leads to the eigenvalue problem for a quadratic pencil operator

$$u - \tau K_1 u + \tau^2 K_2 u = 0, \quad u \in H^2_0(D)$$

with selfadjoint compact operators $K_1 = T^{-1/2} T_1 T^{-1/2}$ and $K_2 = T^{-1/2} T_2 T^{-1/2}$ where

$$(Tu, \varphi)_{H^2(D)} = \int_D \frac{1}{n-1} \Delta u \Delta \varphi \, dx \quad \text{coercive}$$

$$(T_1 u, \varphi)_{H^2(D)} = -\int_D \frac{1}{n-1} (\Delta u \varphi + nu \Delta \varphi) \, dx$$

$$(T_2 u, \varphi)_{H^2(D)} = \int_D \frac{n}{n-1} u \varphi \, dx \quad \text{non-negative}.$$
Transmission Eigenvalues

The transmission eigenvalue problem can be transformed to the eigenvalue problem in $H_0^2(D) \times H_0^2(D)$

$$\left( \mathbb{K} - \xi \mathbb{I} \right) U = 0,$$

$$U = \begin{pmatrix} u \\ \tau K_2^{1/2} u \end{pmatrix}, \quad \xi := \frac{1}{\tau}$$

for the non-selfadjoint compact operator

$$\mathbb{K} := \begin{pmatrix} K_1 & -K_2^{1/2} \\ K_2^{1/2} & 0 \end{pmatrix}.$$ 

However from here one can see that the transmission eigenvalues form a discrete set with $+\infty$ as the only possible accumulation point.

Note: in the special case of spherically stratified media it is possible to prove existence of complex transmission eigenvalues, Leung-Colton (to appear).
Media with Voids

How to approach the transmission eigenvalue problem. We first consider the case $A = I$.

We assume $n \in L^\infty(D)$, $n > 0$

$n = 1$ in $D_0 \subset D$ and

$n - 1 > 0$ or $1 - n > 0$ in $D \setminus \overline{D_0}$.

Literature:


Cossonniere-Haddar (2011) have investigated this problem for Maxwell’s equation.
Recall the transmission eigenvalue problem, and to fix our ideas assume $n - 1 > 0$

\[
\begin{align*}
\Delta w + k^2 nw &= 0 \quad \text{in} \quad D \\
\Delta v + k^2 v &= 0 \quad \text{in} \quad D \\
w &= v \quad \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D
\end{align*}
\]

Let $u = w - v \in H^2(D)$. Then

\[
\begin{align*}
\left(\Delta + k^2 n\right) \frac{1}{n - 1} \left(\Delta + k^2\right) u &= 0 \quad \text{in} \quad D \setminus \overline{D_0} \\
\left(\Delta + k^2\right) u &= 0 \quad \text{in} \quad D_0,
\end{align*}
\]

and

\[
\begin{align*}
u &= 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial D.
\end{align*}
\]
Transmission Eigenvalue Problem

\[ V_0(D, D_0, k) := \{ u \in H^2_0(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0 \}. \]

The transmission eigenvalue problem reads, for \( \psi \in V_0(D, D_0, k) \)

\[
\int_{D \setminus D_0} \frac{1}{n-1} (\Delta + k^2) u (\Delta + k^2) \bar{\psi} \, dx + k^2 \int_{D \setminus D_0} (\Delta u + k^2 u) \bar{\psi} \, dx = 0
\]

It is possible to obtain lower bounds for transmission eigenvalues.

\[
\int_{D \setminus D_0} \frac{1}{n-1} |\Delta u + k^2 nu|^2 \, dx - k^4 \int_{D \setminus D_0} n|u|^2 \, dx + k^2 \int_{D \setminus D_0} |\nabla u|^2 \, dx
\]

\[
- k^4 \int_{D_0} |u|^2 \, dx + k^2 \int_{D_0} |\nabla u|^2 \, dx = 0
\]

Then \( u = 0 \) as long as \( k^2 < \frac{\lambda_1(D)}{\sup_D n} \), where \( \lambda_1(D) \) is the first Dirichlet eigenvalue of \( -\Delta \) in \( D \).
Transmission Eigenvalue Problem

\[ A_k(u, \psi) + B_k(u, \psi) = 0 \text{ for all } \psi \in V_0(D, D_0, k). \]

- \( A_k : V_0(D, D_0, k) \rightarrow V_0(D, D_0, k) \) is self-adjoint and positive definite.
- \( B_k : V_0(D, D_0, k) \rightarrow V_0(D, D_0, k) \) is self-adjoint and compact.

\( k \) is a transmission eigenvalue if and only if the operator

\[ A_k + B_k \quad \text{or} \quad I_k + A_k^{-1/2} B_k A_k^{-1/2} : V_0(D, D_0, k) \rightarrow V_0(D, D_0, k) \]

has a nontrivial kernel where \( I_k \) is the identity operator on \( V_0(D, D_0, k) \).

Obviously, a transmission eigenvalue has finite multiplicity.
Transmission Eigenvalue Problem

To avoid dealing with function spaces depending on $k$ we introduce the orthogonal projection operator $P_k$ from $H^2_0(D)$ onto $V_0(D, D_0, k)$ and the corresponding injection $R_k : V_0(D, D_0, k) \to H^2_0(D)$. Then $k$ is a transmission eigenvalue if and only if the operator

$$I + T_k : H^2_0(D) \to H^2_0(D)$$

has nontrivial kernel, where $T_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ is compact.

- **Discreteness**: Unfortunately $P_k$ is not analytic in a neighborhood of real axis. We build an analytic operator that mimics the effect of projection which allows us to build an analytic extension of $A_k$ and $B_k$ on $H^2_0(D)$. The result follows for the Analytic Fredholm Theory.

- **Existence**: Consider the auxiliary eigenvalue problem

$$\left( I + \lambda(k) T_k \right) u = 0 \quad \text{on} \quad H^2_0(D)$$

and find solutions to $\lambda(k) = 1$. 
Transmission Eigenvalues

\[(\mathbb{I} + \lambda(k)\mathbb{T}_k)u = 0 \quad \text{on} \quad H^2_0(D)\]

For a fixed \( k > 0 \) there exists an increasing sequence of eigenvalues \( \lambda_j(k)_{j \geq 1} \) such that \( \lambda_j(k) \to +\infty \) as \( j \to \infty \).

Thanks to max-min principle \( \lambda_j(k) \) depend continuously on \( k \in [0, \infty) \).

Hence, if there exists two positive constants \( k_0 > 0 \) and \( k_1 > 0 \) such that

- \( \mathbb{I} + \mathbb{T}_{k_0} \) is positive on \( H^2_0(D) \),
- \( \mathbb{I} + \mathbb{T}_{k_1} \) is non positive on a \( m \) dimensional subspace of \( H^2_0(D) \)

then each of the equations \( \lambda_j(k) = 1 \) for \( j = 1, \ldots, m \), has at least one solution in \( [k_0, k_1] \) meaning that there exists \( m \) transmission eigenvalues (counting multiplicity) within the interval \( [k_0, k_1] \).
There exists an infinite set of real transmission eigenvalues accumulating at $+\infty$.

If $k_1(D_0, n(x))$ is the first eigenvalue, then for a fixed $D$ we have:

- The Faber Krahn inequality
  
  $$0 < \frac{\lambda_1(D)}{\sup_D n} \leq k(D_0, n(x)).$$

- Monotonicity with respect to the index of refraction
  
  $$k_1(D_0, n(x)) \leq k_1(D_0, \tilde{n}(x)), \quad \tilde{n}(x) \leq n(x).$$

- Monotonicity with respect to voids
  
  $$k_1(D_0, n(x)) \leq k_1(\tilde{D}_0, n(x)), \quad D_0 \subset \tilde{D}_0.$$ 

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in $D$.

Similar results can be obtained for $1 - n > 0$. 


Numerical Example: Media with Voids


The sphere with radius 1 and $N = 4I$

$k_1 = 3.16$
The sphere with radius 1, $N = 4I$ containing a cubic cavity $k_1 = 3.33$, i.e. $k_1$ is shifted to the right.
The Case of $n - 1$ Changing Sign

Recently, progress has been made in the case of the contrast $n - 1$ changing sign inside $D$ with state of the art result by Sylvester, SIMA (2012). Roughly speaking he shows that transmission eigenvalues form a discrete (possibly empty) set provided $n - 1$ has fixed sign only in a neighborhood of $\partial D$. There are two aspects in the proof:

- **Fredholm property.** Sylvester considers the problem in the form

  \[
  \Delta u + k^2 nu = k^2 (n - 1) v, \quad \Delta v + k^2 v = 0, \quad u \in H^2_0(D), \; v \in H^1(D)
  \]

  and uses the concept of upper-triangular compact operators. This property can also be obtained via variational formulation or integral equation formulation, Haddar-Cossoniere, (to appear).

- **Find a $k$ that is not a transmission eigenvalues.** This requires careful estimates for the solution inside $D$ in terms of its values in a neighborhood of $\partial D$.

The existence of transmission eigenvalues under such weaker assumptions is still open.
Cakoni-Colton-Haddar, Inverse Problems (2012) has initiated the study of the existence transmission eigenvalues for the case of absorbing media.

\[
\begin{align*}
\Delta w + k^2 \left( \epsilon_1 + i \frac{\gamma_1}{k} \right) w &= 0 \quad \text{in } D \\
\Delta v + k^2 \left( \epsilon_0 + i \frac{\gamma_0}{k} \right) v &= 0 \quad \text{in } D
\end{align*}
\]

where \( \epsilon_0 \geq \alpha_0 > 0, \epsilon_1 \geq \alpha_1 > 0, \gamma_0 \geq 0, \gamma_1 \geq 0 \) are bounded functions.

For the corresponding spherically stratified case there exits infinitely many (complex) transmission eigenvalues.

It is possible for real transmission eigenvalues to exit for some combinations of media and background.
Absorbing-Dispersive Media

In the general case:

- The set of transmission eigenvalues $k \in \mathbb{C}$ in the right half plane is discrete, provided $\varepsilon_1(x) - \varepsilon_0(x) > 0$.

- Using the stability of a finite set of eigenvalues for closed operators one can show that if $\sup\gamma_0 + \gamma_1$ is small enough there exists at least $\ell > 0$ transmission eigenvalues each in a small neighborhood of the first $\ell$ real transmission eigenvalues corresponding to $\gamma_0 = \gamma_1 = 0$.

- For the case of $\varepsilon_0, \varepsilon_1, \gamma_0, \gamma_1$ constant, we have identified eigenvalue free zones in the complex plane.

The existence of transmission eigenvalues for general media if absorption is present is still open.
The corresponding transmission eigenvalue problem is to find \( v, w \in H^1(D) \) such that

\[
\begin{align*}
\nabla \cdot A \nabla w + k^2 n w &= 0 \quad \text{in} \quad D \\
\Delta v + k^2 v &= 0 \quad \text{in} \quad D \\
w &= v \quad \text{on} \quad \partial D \\
\nu \cdot A \nabla w &= \nu \cdot \nabla v \quad \text{on} \quad \partial D.
\end{align*}
\]

This transmission eigenvalue problem has a more complicated nonlinear structure than quadratic.

Existence of Transmission Eigenvalues

Set $u = w - v \in H^1_0(D)$. Find $v = v_u$ by solving a Neuman type problem: For every $\psi \in H^1(D)$

$$
\int_D (A - I) \nabla v \cdot \nabla \psi - k^2(n - 1)v \psi \, dx = \int_D A \nabla u \cdot \nabla \psi - k^2 nu \psi \, dx.
$$

Having $u \to v_u$, we require that $v := v_u$ satisfies $\Delta v + k^2 v = 0$.

Thus we define $\mathbb{I}_k : H^1_0(D) \to H^1_0(D)$

$$(\mathbb{I}_k u, \phi)_{H^1_0(D)} = \int_D \nabla v_u \cdot \nabla \phi - k^2 v_u \cdot \phi \, dx, \quad \phi \in H^1_0(D).$$

Then the transmission eigenvalue problem is equivalent to

$$\mathbb{I}_k u = 0 \quad \text{in} \quad H^1_0(D)$$

which can be written

$$(\mathbb{I} + \mathbb{I}_0^{-1/2} \mathbb{C}_k \mathbb{I}_0^{-1/2}) u = 0 \quad \text{in} \quad H^1_0(D)$$

$\mathbb{I}_0$ self-adjoint positive definite and $\mathbb{C}_k$ self-adjoint compact.
Existence of Transmission Eigenvalues

- If \( n(x) \equiv 1 \) and the contrast \( A - I \) is either positive or negative in \( D \) then there exists an infinite discrete set of real transmission eigenvalues accumulating at \( +\infty \).

- If the contrasts \( A - I \) and \( n - 1 \) have the opposite fixed sign, then there exists an infinite discrete set of real transmission eigenvalues accumulating at \( +\infty \).

- If the contrasts \( A - I \) and \( n - 1 \) have the same fixed sign, then there exits at least one real transmission eigenvalue providing that \( n \) is small enough.
The strongest result on the discreteness of transmission eigenvalues for this problem is due to Bonnet Ben Dhia - Chesnel - Haddar, Comptes Rendus Math. (2011) (using the concept of $\mathsf{T}$- coercivity).

In particular, the discreteness of transmission eigenvalues is proven under either one of the following assumptions (weaker than for the existence):

- Either $A - I > 0$ or $A - I < 0$ in $D$, and $\int_D (n - 1) \, dx \neq 0$ or $n \equiv 1$.
- The contrasts $A - I$ and $n - 1$ have the same fixed sign only in a neighborhood of the boundary $\partial D$. 
**Numerical Example: Homogeneous Anisotropic Media**

Take \( n = 1 \). Find an isotropic-homogenous media \( a_0 \) that has the first transmission eigenvalue the same as the (measured) first transmission eigenvalue for the unknown anisotropic media.

We consider \( D \) to be the unit square \([-1/2, 1/2] \times [-1/2, 1/2]\) and

\[
A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \quad A_2 = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix} \quad A_{2r} = \begin{pmatrix} 7.4136 & -0.9069 \\ -0.9069 & 6.5834 \end{pmatrix}
\]

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Eigenvalues ( a_<em>, a^</em> )</th>
<th>Predicted ( a_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{iso} )</td>
<td>4, 4</td>
<td>4.032</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>2, 8</td>
<td>5.319</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>6, 8</td>
<td>7.407</td>
</tr>
<tr>
<td>( A_{2r} )</td>
<td>6, 8</td>
<td>6.896</td>
</tr>
</tbody>
</table>

*Cakoni-Colton-Monk-Sun, Inverse Problems, (2010)*
Examples for Anisotropic Maxwell’s Equations. $D$ is a sphere of radius 1. The matrix $N = \varepsilon / \varepsilon_0$ is relative electric permittivity. The relative magnetic permeability is one.

Perturbation of $N = 16I$

Perturbation of $N = 5I$
We first compute the transmission eigenvalues for anisotropic $N$ from measured scattering data. Then compute the isotropic $n_0$ that has the same first transmission eigenvalue as the measured one.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$k_{1,D,N(x)}$</th>
<th>$n_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>diag([15.5, 16, 16.5])</td>
<td>1.163</td>
<td>16.33</td>
</tr>
<tr>
<td>diag([15, 16, 17])</td>
<td>1.151</td>
<td>16.65</td>
</tr>
<tr>
<td>diag([16, 16, 17])</td>
<td>1.161</td>
<td>16.38</td>
</tr>
<tr>
<td>diag([16, 16, 16.5])</td>
<td>1.161</td>
<td>16.38</td>
</tr>
<tr>
<td>diag([16, 16, 17])</td>
<td>1.146</td>
<td>16.77</td>
</tr>
</tbody>
</table>
The same procedure can be carried out at lower $N$ as well (the lowest transmission eigenvalue increases)

<table>
<thead>
<tr>
<th>$N$</th>
<th>$k_{1,D,N(x)}$</th>
<th>$n_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>diag([4.5, 5, 5.5])</td>
<td>2.442</td>
<td>5.339</td>
</tr>
<tr>
<td>diag([4, 5, 6])</td>
<td>2.302</td>
<td>5.631</td>
</tr>
<tr>
<td>diag([5, 5, 5.5])</td>
<td>2.410</td>
<td>5.397</td>
</tr>
<tr>
<td>diag([5, 5, 6])</td>
<td>2.245</td>
<td>5.778</td>
</tr>
</tbody>
</table>
Open Problems

- Can the existence of real transmission eigenvalues for non-absorbing media be established if the assumptions on the sign of the contrast are weakened?
- Do complex transmission eigenvalues exist for general non-absorbing media?
- Do real transmission eigenvalues exist for absorbing media?
- What would the necessary conditions be on the contrasts that guaranty the discreteness of transmission eigenvalues?
- Can Faber-Krahn type inequalities be established for the higher eigenvalues?
- Can an inverse spectral problem be developed for the general transmission eigenvalue problem? (Completeness of eigen-solutions?)