A Bellman approach for optimal control problems on multi-domains

Ariela Briani

Joint work with G. Barles & E. Chasseigne
Model problem

I am on the beach and I want to take my boat in minimal time:

- How do I choose my trajectory? How do I define my trajectory when I ”swim and walk”? 

- How do I take into account the sea when I am on the beach and viceversa?

Goal: To modellize optimal control problem with ”different dynamics and/or costs in different domains”
Infinite Horizon, 2-domains, control problems

Dynamic : \( b_1 \)
\[ \Omega_1 := \{ x \in \mathbb{R}^N : x_N > 0 \} \]

Running cost : \( l_1 \)
\[
\begin{align*}
\min \{ H_1, H_2 \} & \leq 0 \\
\max \{ H_1, H_2 \} & \geq 0
\end{align*}
\]

Dynamic : \( b_2 \)
\[ \Omega_2 := \{ x \in \mathbb{R}^N : x_N < 0 \} \]

Running cost : \( l_2 \)
\[ H_2 = 0 \text{ in } \Omega_2 \]

\( (\text{Same constant discount factor in } \Omega_1 \text{ and } \Omega_2 : \lambda) \)

Questions : How to define the dynamic and running cost on \( \mathcal{H} := \{ x \in \mathbb{R}^N : x_N = 0 \} \)? What is the Bellman problem satisfied by the value function(s)? What are the right viscosity inequalities to be satisfied on \( \mathcal{H} \)?
Related works

– P. Dupuis (91-92) : dimension 1, calculus of variations with discontinuous integrand

– Bressan and Hong, Wolenski : optimal control problem on stratified domains.

– Soravia, Garavello and Soravia, De Zan and Soravia : problems with discontinuities but with a special structure of the discontinuities

– Camilli and Siconolfi : $L^\infty$-framework, but special equations

Standard assumptions: Regularity and boundedness for $b_i, l_i \ (i = 1, 2)$; convexity and controllability for the sets $\{(b_i(x, \alpha_i), l_i(x, \alpha_i)) : \alpha_i \in A_i\}$.

Controlled trajectories: solve the differential inclusion

$$\dot{X}_{x_0}(t) \in \mathcal{B}(X_{x_0}(t)) \quad t \in (0, +\infty) ; \quad X_{x_0}(0) = x_0$$

where

$$\mathcal{B}(x) := \begin{cases} B_1(x) & \text{if } x_N > 0 , \\ B_2(x) & \text{if } x_N < 0 , \\ \overline{\text{co}}(B_1(x) \cup B_2(x)) & \text{if } x_N = 0 , \end{cases}$$

the notation $\overline{\text{co}}(E)$ referring to the convex closure of the set $E \subset \mathbb{R}^N$. 
Theorem: (true without the controllability assumption)

(i) For each \( x_0 \in \mathbb{R}^N \), there exists a solution of the differential inclusion.

(ii) For each solution \( X_{x_0}(\cdot) \), there exists a control \( a(\cdot) = (\alpha_1(\cdot), \alpha_2(\cdot), \mu(\cdot)) \) such that

\[
\dot{X}_{x_0}(t) = b_1(X_{x_0}(t), \alpha_1(t)) \mathbb{1}_{\{X_{x_0}(t) \in \Omega_1\}} \\
+ b_2(X_{x_0}(t), \alpha_2(t)) \mathbb{1}_{\{X_{x_0}(t) \in \Omega_2\}} \\
+ b_\mathcal{H}(X_{x_0}(t), a(t)) \mathbb{1}_{\{X_{x_0}(t) \in \mathcal{H}\}}.
\]

where

\[
b_\mathcal{H}(x, (\alpha_1, \alpha_2, \mu)) := \mu b_1(x, \alpha_1) + (1 - \mu) b_2(x, \alpha_2).
\]

(iii) If \( e_N = (0, \cdots, 0, 1) \), then

\[
b_\mathcal{H}(X_{x_0}(t), a(t)) \cdot e_N = 0 \quad \text{a.e. on } \{X_{x_0,N}(t) = 0\}.
\]
Running cost: define

\[ \ell(X_{x_0}(t), a(t)) := l_1(X_{x_0}(t), \alpha_1(t)) \mathbb{1}\{x_{x_0}(t) \in \Omega_1\} \]
\[ + l_2(X_{x_0}(t), \alpha_2(t)) \mathbb{1}\{x_{x_0}(t) \in \Omega_2\} \]
\[ + l_H(X_{x_0}(t), a(t)) \mathbb{1}\{x_{x_0}(t) \in \mathcal{H}\} \cdot \]

where

\[ l_H(x, a) = l_H(x, (\alpha_1, \alpha_2, \mu)) := \mu l_1(x, \alpha_1) + (1 - \mu) l_2(x, \alpha_2) \cdot \]

Cost:

\[ J(x_0; (X_{x_0}, a)) := \int_0^\infty \ell(X_{x_0}(t), a) e^{-\lambda t} dt \]
Regular and Singular Strategies on $\mathcal{H}$, the dynamic is:

$$b_{\mathcal{H}}(x, a) = \mu b_1(x, \alpha_1) + (1-\mu)b_2(x, \alpha_2), \quad b_{\mathcal{H}}(x, a) \cdot e_N = 0$$

The regular strategies ("both pushes to be on $\mathcal{H}$"):

$$b_1(x, \alpha_1) \cdot e_N \leq 0 \quad \text{and} \quad b_2(x, \alpha_2) \cdot e_N \geq 0.$$ 

The singular strategies ("both pull so we stay on $\mathcal{H}$"):

$$b_1(x, \alpha_1) \cdot e_N > 0 \quad \text{and} \quad b_2(x, \alpha_2) \cdot e_N < 0.$$ 

Therefore two "natural" value functions can be defined

$$U^-(x_0) := \inf_{(X_{x_0}, a) \in \mathcal{T}_{x_0}} J(x_0; (X_{x_0}, a))$$

$\mathcal{T}_{x_0}$: with regular and singular strategies on $\mathcal{H}$

$$U^+(x_0) := \inf_{(X_{x_0}, a) \in \mathcal{T}_{x_0}^{\text{reg}}} J(x_0; (X_{x_0}, a))$$

$\mathcal{T}_{x_0}^{\text{reg}}$: without the singular strategies on $\mathcal{H}$

NB : $U^- \leq U^+$ in $\mathbb{R}^N$. 
Our results:

**Theorem:**
The bounded, Lipschitz continuous value functions $U^-$ and $U^+$ are viscosity solutions of the “natural” Hamilton-Jacobi-Bellman problem

$$\begin{cases}
H_1(x, u, Du) = 0 & \text{in } \Omega_1 \\
H_2(x, u, Du) = 0 & \text{in } \Omega_2 \\
\min \{H_1(x, u, Du), H_2(x, u, Du)\} \leq 0 & \text{on } \mathcal{H} \\
\max \{H_1(x, u, Du), H_2(x, u, Du)\} \geq 0 & \text{on } \mathcal{H}.
\end{cases}$$

Moreover:

(i) $U^-$ is the minimal supersolution and solution.

(ii) $U^+$ is the maximal subsolution and solution.
Which condition is verified by $U^−$ and $U^+$ on $H$?

$x′ \mapsto U^−(x′, 0)$ verifies

$$H_T(x, u, D_Hu) \leq 0 \quad \text{on } H,$$

where $D_Hu$ is (in fact) $D_{x′}u$,

$$H_T(x, u, p′) = \sup_{A_0(x)} \{-b_H(x, a) \cdot (p′, 0) + \lambda u - l_H(x, a)\}.$$

and $A_0(x)$ is the set of control $a = (\alpha_1, \alpha_2, \mu)$ such that $b_H(x, (\alpha_1, \alpha_2, \mu)) \cdot e_N = 0$.

$x′ \mapsto U^+(x′, 0)$ satisfies

$$H^\text{reg}_T(x, u, D_Hu) \leq 0 \quad \text{on } H,$$

where $H^\text{reg}_T$ is given by the same definition as $H_T$, $A_0(x)$ being replaced by $A^\text{reg}_0(x)$ consisting in elements of $A_0(x)$ satisfying $b_1(x, \alpha_1) \cdot e_N \leq 0$ and $b_2(x, \alpha_2) \cdot e_N \geq 0$.

(We do not allow singular strategies).
Moreover any subsolution of the system satisfies

\[ H^\text{reg}_T(x, w, D_H w) \leq 0 \quad \text{on } \mathcal{H} \]

Thus this inequality is encoded in the original problem and not an additional property: this is not the case for the singular strategies!

With the additional \( H_T \leq 0 \)-inequality, we have a uniqueness result:

**Theorem (Strong Comparison Result):** Assume that \( u \) and \( v \) are respectively bounded sub and supersolution of

\[
\begin{cases}
H_1(x, w, Dw) = 0 & \text{in } \Omega_1 \\
H_2(x, w, Dw) = 0 & \text{in } \Omega_2 \\
\min \{ H_1(x, w, Dw), H_2(x, w, Dw) \} \leq 0 & \text{on } \mathcal{H} \\
\max \{ H_1(x, w, Dw), H_2(x, w, Dw) \} \geq 0 & \text{on } \mathcal{H}.
\end{cases}
\]

and that

\[ H_T(x, u, D_H u) \leq 0 \quad \text{on } \mathcal{H} \]

Then \( u \leq v \) in \( \mathbb{R}^N \).
thanks for your attention.