The heat equation in a non-cylindrical domain governed by a subdifferential inclusion

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Outline

1 The heat equation in time-varying domains.
2 Morphological shape equations: a way to describe evolving sets.
3 The heat equation in a time-varying domain described by a subdifferential inclusion.
4 Conclusions.
The heat equation in time-varying domains.

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\[ u_t(x, t) - \text{div} \left( a(x) \nabla u(x, t) \right) = f(x, t), \quad \text{in } Q \]
\[ u = 0, \quad \text{on } \Sigma \]
\[ u(x, 0) = u_0(x), \quad x \in \Omega \]

(CP)

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Describing the evolution of \( \Omega_t \): Velocity method

Usually \( \Omega_t \) is assumed to be generated by the flow of a nonautonomous vector field

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That is $\Omega_t = T_t(\Omega_0)$, where

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Cannarsa, Da Prato & Zolésio (1989, 1990)
Zolésio (2004)
Burdzy, Chen & Sylvester (2004)

etc…
Describing the evolution of $\Omega_t$: Velocity method

$\Omega_0 \rightarrow \Omega_t = T_t(\Omega_0)$

$T_t(x)$

$\mathbf{V}$
The classical procedure to solve (NCP) is:

✓ Show that $T_t$ is a diffeomorphism for any $t$, Lipschitz w.r. to $t$.
✓ Use $T_t$ to transform the heat equation into a parabolic one with variable coefficients defined in the reference cylinder $\Omega_0 \times ]0, T[$.
✓ Establish the existence of solution $\omega(x, t)$ of the parabolic problem.
✓ The map $u(x, t) = \omega(T_t^{-1}(x), t)$ provides the solution of the original non-cylindrical problem.
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Límaco, Medeiros & Zuazua (2002)
Aim

Focussing on the evolution of the spatial domain:

- Can we consider non-cylindrical problems for domains evolving in a more general way?
- Is it possible to solve problems where the velocity (in some sense) of the domain depends on its global shape?
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Let $\mathbb{R}^N$ be endowed with the usual (Euclidean) norm $| \cdot |$ and let $\mathcal{K}(\mathbb{R}^N)$ be the family of all its nonempty compact subsets.

Equipped with the Hausdorff distance,

$$d_H(K, M) := \max \left( \sup_{x \in K} d_M(x), \sup_{z \in M} d_K(z) \right),$$

$\mathcal{K}(\mathbb{R}^N)$ is a complete separable metric space, also satisfying that closed balls are compact. However, $\mathcal{K}(\mathbb{R}^N)$ has not a linear (vector) structure at all!
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Aubin (1999)
Delfour & Zolésio (2001)
Lorenz (2010)
Let $C^{0,1}(\mathbb{R}^N; \mathbb{R}^N)$ be the family of all Lipschitz vector fields.

For $V \in C^{0,1}(\mathbb{R}^N; \mathbb{R}^N)$, $\vartheta_V(t, K) = \{ T_t(x) : x \in K \}$ is the reachable set at time $t$ associated to the solutions

$$\frac{\partial T_t(x)}{\partial t} = V(T_t(x)), \quad T_0(x) = x$$

starting from $K \subset \mathbb{R}^N$.

The map $h \mapsto \vartheta_V(h, K)$ provides a curve (a shape transition) on $\mathcal{K}(\mathbb{R}^N)$. These transitions will play the role of “directions” in $\mathcal{K}(\mathbb{R}^N)$.

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$$\frac{d_H(\vartheta_V(t + h, K), \vartheta_V(h, \vartheta_V(t, K)))}{h} \to 0, \quad \text{as } h \to 0,$$

we can see the field $V$ as the “velocity at time $t$” of the tube $\vartheta_V(\cdot, K)$.
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✓ Given a tube $K : I \subset \mathbb{R} \rightarrow \mathcal{C}(\mathbb{R}^N)$, it is said that $V \in C^{0,1}(\mathbb{R}^N; \mathbb{R}^N)$ belongs to the *shape mutation* of $K(\cdot)$ at $t$ in the forward direction if

$$
\lim_{h \rightarrow 0^+} \frac{d_H(\vartheta_V(h, K(t)), K(t+h))}{h} = 0
$$

Then we will write $V \in \dot{K}(t)$.

✓ The set $\dot{K}(t) \subset C^{0,1}(\mathbb{R}^N; \mathbb{R}^N)$ can be regarded as the “velocity” of $K(\cdot)$ at time $t$.

✓ A map $V : I \subset \mathbb{R} \rightarrow C^{0,1}(\mathbb{R}^N; \mathbb{R}^N)$ will be a “shape primitive” of $K(\cdot)$ if for any $t$ the field $V(t)$ belongs to its shape mutation at $t$. We will write

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Remark (Doyen, 1995). The shape mutation of a tube is usually set-valued. For instance if $K(t) = B$ is constant and equal to the closed unit ball in $\mathbb{R}^2$, it is clear that for every $t$,

$$0, V(x, y) = (-y, x) \in \text{int } K(t)$$
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$$0, V(x, y) = (-y, x) \in \partial K(t)$$

Remark. Every field $V : I \subset \mathbb{R} \rightarrow C^{0,1}(\mathbb{R}^N; \mathbb{R}^N)$ is a shape primitive of the reachable tube $t \sim \vartheta_V(t, K), K \in \mathcal{K}(\mathbb{R}^N)$. 

Shape mutations: a tool to define the evolving velocity of sets
We are ready to define *morphological shape equations* as a generalization of ordinary differential equations governing the evolution of tubes.

For a map \( V : \mathbb{R}_+ \times \mathcal{K}(\mathbb{R}^N) \rightarrow C^{0,1}(\mathbb{R}^N, \mathbb{R}^N) \), a solution of the *morphological shape equation*

\[
\circ K(t) \ni V(t, K(t))\cdot
\]

on an interval \( I \subset \mathbb{R}_+ \) will be a compact-valued Lipschitz tube, \( K(\cdot) \) such that, for all \( t \in I \), \( V(t, K(t)) \) belongs to \( \circ K(t) \), that is

\[
\lim_{h \to 0^+} \frac{d_{\mathcal{H}}(\partial_{V(t,K(t))}(h, K(t)), K(t+h))}{h} = 0
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Solutions of (♠) satisfy the recurrence law

\[
K(t) = \{ x(t) : \dot{x}(s) = V(s, K(s))(x(s)), \ x(0) \in K(0) \}
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$$K(t) \ni \mathbf{V}(t, K(t))(\cdot)$$ \hspace{1cm} (♠)

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Morphological shape equations

Remark. The reachable tube $t \mapsto \vartheta_{V}(t, K) = \{ T_{t}(x) : x \in K \}$ with

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\begin{align*}
\frac{\partial T_{t}(x)}{\partial t} &= V(t, T_{t}(x)) \\
T_{0}(x) &= x
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is the solution of the morphological shape equation $\hat{K}(t) \ni V(t, \cdot)$. Thus the family of solutions of morphological shape equations contains all the tubes described by flows of vector fields.
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Remark. This inclusion is strict. Indeed evolutions where the velocity depends on the global shape are allowed. For instance taking

$$V(t, K) = W(t) + \varphi(K)F$$

with $W(t), F \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$ and

$$\varphi(K) = \begin{cases} 
0, & \text{if } d_H(K, M) \geq \delta \\
\delta - d_H(K, M), & \text{otherwise}
\end{cases}$$
Existence theorem for morphological shape equations

**Theorem 1 (Cauchy-Lipschitz for shape equations, Aubin, 1999)**

Let $V : \mathbb{R}_+ \times \mathcal{K}([0, T]) \to C^{0,1}(\mathbb{R}_+; \mathbb{R}^N)$ be continuous with respect to the first variable $t$, $\lambda$-Lipschitz with respect to the second one $K$, that is,

$$\|V(t, K) - V(t, B)\|_\infty \leq \lambda d_H(K, B)$$

and satisfying

$$\alpha := \sup_{0 < t < T, K \in \mathcal{K}(\mathbb{R}^N)} \sup_{x \neq y} \left( \frac{|V(t, K)(x) - V(t, K)(y)|}{|x - y|} \right) < \infty$$

then for any $K \in \mathcal{K}(\mathbb{R}^N)$ there exists a unique solution of

$${^\circ}K(t) \ni V(t, K(t))$$

satisfying $K(0) = K$. 

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Existence theorem for morphological shape equations

Theorem 1 (Cauchy-Lipschitz for shape equations, Aubin, 1999)

Let $V : \mathbb{R}_+ \times \mathcal{K}(\mathbb{R}^N) \rightarrow C^{0,1}(\mathbb{R}^N; \mathbb{R}^N)$ be continuous with respect to the first variable $t$, $\lambda$-Lipschitz with respect to the second one $K$, that is,

$$\|V(t, K) - V(t, B)\|_\infty \leq \lambda d_H(K, B)$$

and satisfying

$$\alpha := \sup_{0 < t < T, K \in \mathcal{K}(\mathbb{R}^N)} \sup_{x \neq y} \left( \frac{|V(t, K)(x) - V(t, K)(y)|}{|x - y|} \right) < \infty$$

then for any $K \in \mathcal{K}(\mathbb{R}^N)$ there exists a unique solution of

$$\overset{\circ}{K}(t) \ni V(t, K(t))(\cdot)$$

satisfying $K(0) = K$. Moreover, if $K(\cdot)$ and $B(\cdot)$ are solutions starting from $K$ and $B$ respectively, then

$$\forall t, \quad d_H(K(t), B(t)) \leq e^{(\alpha + \lambda)t} d_H(K, B)$$
Existence theorem for morphological shape equations

Let $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ be convex and l.s.c.
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Let $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ be convex and l.s.c.

By means of its subdifferential $\partial \phi$ we can perturb the equation

$$K(t) \ni V(t, K(t))(\cdot)$$

(♠)
Existence theorem for morphological shape equations

Let $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ be convex and l.s.c.

By means of its subdifferential $\partial \phi$ we can perturb the equation to get the morphological shape inclusion

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$$\hat{K}(t) \cap (-\partial \phi(\cdot) + \mathbf{V}(t, K(t))(\cdot)) \neq \emptyset$$

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Theorem 2

Assuming hypotheses of Theorem 1, for any $K \in \mathcal{K}(\mathbb{R}^N)$, there exists a unique solution of (♣) with $K(0) = K$. 
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**Theorem 2**

Assuming hypotheses of Theorem 1, for any $K \in \mathcal{K}(\mathbb{R}^N)$, there exists a unique solution of (♣) with $K(0) = K$. Furthermore

$$K(t) = \{x(t) : \dot{x}(s) \in -\partial \phi(x(s)) + \mathbf{V}(s, K(s))(x(s)), \; x(0) \in K\}$$
Existence theorem for morphological shape equations

Let \( \phi : \mathbb{R}^N \to \mathbb{R} \) be convex and l.s.c.

By means of its subdifferential \( \partial \phi \) we can perturb the equation to get the morphological shape inclusion

\[
\circ \ K(t) \cap (-\partial \phi(\cdot) + V(t, K(t))(\cdot)) \neq \emptyset \tag{♣}
\]

**Theorem 2**

*Assuming hypotheses of Theorem 1, for any \( K \in \mathcal{K}([\mathbb{R}^N]) \), there exists a unique solution of (♣) with \( K(0) = K \). Furthermore

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**Remark.** Moreau envelope \( \phi_\mu(x) = \inf_{y \in \mathbb{R}^N} \left( \phi(y) + \frac{1}{2\mu} \|y - x\|^2 \right) \) allows to prove this result.
Outline

1. The heat equation in time-varying domains.
2. Morphological shape equations: a way to describe evolving sets.
3. The heat equation in a time-varying domain described by a subdifferential inclusion.
The problem

\[
\begin{aligned}
&u_t(x, t) - \text{div} \left( a(x) \nabla u(x, t) \right) = f(x, t), \quad \text{in } Q \\
u = 0, \quad \text{on } \Sigma \\
u(x, 0) = u_0(x), \quad x \in \Omega_0
\end{aligned}
\]

(NCP)

where \( K(t) = \overline{\Omega_t} \) satisfyies

\[
K(t) \cap (-\partial \phi(\cdot) + V(t, K(t))(\cdot)) \neq \emptyset
\]

(♣)
Main result

\[ u_t(x, t) - \text{div} \left( a(x) \nabla u(x, t) \right) = f(x, t), \quad \text{in } Q \]
\[ u = 0, \quad \text{on } \Sigma \]
\[ u(x, 0) = u_0(x), \quad x \in \Omega_0 \]

\[ \left\{ \begin{array}{l}
\end{array} \right. \quad \text{(NCP)} \]

where \( K(t) = \overline{\Omega}_t \) satisfies

\[ \dot{K}(t) \cap (-\partial \phi(\cdot) + \mathbf{V}(t, K(t))(\cdot)) \neq \emptyset \quad \text{(♣)} \]

**Theorem 3**

For any initial states \( u_0 \in L^2(\Omega_0) \), \( \Omega_0 \subset \mathbb{R}^N \) a nonempty open bounded set, there exists a unique solution of the problem, i.e. a tube \( K(\cdot) \in C^{0,1}(0, T; \mathcal{K}(\mathbb{R}^N)) \) and a map

\[ u \in L^2(0, T; H^1_0(\Omega_t)) \cap C([0, T]; L^2(\Omega_t)) \]

satisfying the heat equation in a weak sense.
Conclusions

✓ Morphological shape equations provide time-evolving families of sets in a more general way than the “flows of velocities”.

✓ It is possible to consider (and solve) noncylindrical problems associated with these type of set evolutions for the heat equation, even when a subdifferential perturbation is considered.

✓ This scheme could be appropriate for problems involving different kind of PDEs.

✓ A challenging task is to consider evolutions associated to set-valued maps, where the topology of domains could be modified along time, or having velocities depending on the temperature.
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Thanks for attention!