Controllability of the bilinear Schrödinger equation via nondegenerate connectedness chains

Mario Sigalotti
INRIA Saclay, Team GECO and CMAP, Paris, France

joint work with
U. Boscain, CMAP/CNRS, Paris, France
M. Caponigro, Rutgers, USA
T. Chambriorn, IECN/UHP, Nancy, France

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Physical model: the controlled Schrödinger equation

Schrödinger equation

\[ i\dot{\psi} = -\Delta \psi + V \psi + uW \psi \]

\( \Omega \) domain of \( \mathbb{R}^d \)
\( \psi(t, x) \) wave function, \( \psi(t, \cdot) \in L^2(\Omega), \|\psi(t, \cdot)\|_2 = 1 \)
\( V : \Omega \to \mathbb{R} \) potential
\( -\Delta + V \) Schrödinger operator
\( u = u(t) \) real-valued control
\( W : \Omega \to \mathbb{R} \) controlled potential
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In both cases we will assume that the spectrum of the Schrödinger operator is discrete (\( \lim_{|x| \rightarrow \infty} V(x) = +\infty \) if \( \Omega = \mathbb{R}^d \))
Examples

**Harmonic oscillator**

\[ i \frac{\partial \psi(x, t)}{\partial t} = \left( -\frac{\partial^2}{\partial x^2} + x^2 + u(t)x \right) \psi(x, t), \quad x \in \mathbb{R} \]

- external field constant with respect to \( x \)
- \( u(t) \) intensity of the external field at time \( t \)
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**Potential well**

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i \frac{\partial \psi(x, t)}{\partial t} = \left(- \frac{\partial^2}{\partial x^2} + u(t)x\right) \psi(x, t), \quad x \in (-1, 1), \quad \psi(\pm1, t) = 0
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Orientation of a linear molecule in the plane

\[ i \frac{\partial \psi(\theta,t)}{\partial t} = \left( -\frac{\partial^2}{\partial \theta^2} + u(t)\cos(\theta) \right) \psi(\theta,t), \quad \theta \in S^1 \]

- \( \theta \) rotational degree of freedom of a linear polarized molecule
- magnetic field pointing in the direction \((0, 1)\)
Controllability results

Negative results

- non-exact controllability in the unit sphere of $L^2(Ω)$ (Ball-Marsden-Slemrod [1982], Turinici [2000], Illner-Lange-Teismann [2006]);

- approximate controllability does not hold for the harmonic oscillator (Mirrahimi-Rouchon [2004]).
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Positive results

- exact controllability in $H^3(\Omega)$ for the potential well (Beauchard [2005], Beauchard-Coron [2006]);
- $L^2$- and $H^s$-approximate controllability by Lyapunov methods (Mirrahimi [2006], Ito-Kunisch [2009], Nersesyan [2009]).
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More than one scalar control (Eberly–Law-like systems)
Adami-Boscain [2005], Bloch-Brockett-Rangan [2006], Ervedoza-Puel [2009].
We will consider control systems of the form

\[ \frac{d\psi}{dt} = A(\psi) + uB(\psi), \quad u \in U \quad (A, B, U) \]

with

- \( H \) complex Hilbert space
- \( U \subset \mathbb{R} \)
Mathematical framework

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- \((\phi_n)_{n \in \mathbb{N}}\) orthonormal basis of \(H\) made of eigenvectors of \(A\)
- \(\phi_n \in D(B)\) for every \(n \in \mathbb{N}\)
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- \( \phi_n \in D(B) \) for every \( n \in \mathbb{N} \)
- for every \( u \in U \), \( A + uB \) skew-adjoint operator on \( H \) (possibly unbounded)

The hypotheses above guarantee that \( \forall u \in U, \ e^{t(A+uB)} : H \to H \) is a well-defined group of unitary transformations.
We call $\psi : t_k \mapsto e^{t_k(A+u_kB)} \circ \cdots \circ e^{t_1(A+u_1B)}(\psi_0)$ the solution of the control system $(A, B, U)$ starting from $\psi_0$ associated to the piecewise constant control $u_1 \chi_{[0,t_1]} + u_2 \chi_{[t_1,t_1+t_2]} + \cdots$
We call \( \psi : t_k \mapsto e^{t_k(A+u_kB)} \circ \cdots \circ e^{t_1(A+u_1B)}(\psi_0) \) the solution of the control system \((A, B, U)\) starting from \( \psi_0 \) associated to the piecewise constant control \( u_1 \chi[0,t_1] + u_2 \chi[t_1,t_1+t_2] + \cdots \).

Notice that such a \( \psi(\cdot) \) satisfies, for every \( n \in \mathbb{N} \) and almost every \( t \in [0, T] \)

\[
\frac{d}{dt} \langle \psi(t), \phi_n \rangle = -\langle \psi(t), (A + u(t)B)\phi_n \rangle.
\]

If \( B \) is bounded and \( u \in L^1([0, T], U) \), then there exists a unique weak (and mild) solution \( \psi \in C([0, T], H) \) which coincides with \( \psi(\cdot) \) when \( u \) is piecewise constant. Moreover, if \( \psi_0 \in D(A) \) and \( u \in C^1([0, T], U) \) then \( \psi \) is differentiable and it is a strong solution [Ball-Marsden-Slemrod]
Approximate controllability

We say that \((A, B, U)\) is **approximately controllable** if for every \(\psi_0, \psi_1 \in S\) and every \(\varepsilon > 0\) there exist \(k \in \mathbb{N}\), \(t_1, \ldots, t_k > 0\) and \(u_1, \ldots, u_k \in U\) such that

\[
\|\psi_1 - e^{t_k(A+u_kB)} \circ \ldots \circ e^{t_1(A+u_1B)}(\psi_0)\| < \varepsilon.
\]
Controllability result

\((\lambda_n)_{n \in \mathbb{N}}\) eigenvalues of \(A\) corresponding to \((\phi_n)_{n \in \mathbb{N}}\).

**Theorem (Boscain, Caponigro, Chambriion, S.)**

Assume that there exists \(S \subset \mathbb{N}^2\) such that

- \(S\) connects \(\mathbb{N}\) (\((j, k)\) connected if \((j, k)\) \(\in S\) + transitivity)
- \(\langle \phi_j, B\phi_k \rangle \neq 0\) for every \((j, k)\) \(\in S\)
- each \(\lambda_j\) is simple
- \((j, k)\) \(\in S\) and \((j, k) \neq (m, l) \in \mathbb{N}^2\) \(\implies |\lambda_j - \lambda_k| \neq |\lambda_m - \lambda_l|\)
  or \(\langle \phi_m, B\phi_l \rangle = 0\).

Then

\[
\frac{d\psi}{dt} = A(\psi) + uB(\psi), \quad u \in [0, \delta],
\]

is approximately controllable for every \(\delta > 0\).

- \(W\) unbounded is allowed
- bounded (arbitrarily small) controls
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- convexification procedure: high-order Galerkyn approximations’ dynamics can mimic low order ones in projections
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- approximate controllability *in modulus* in observed projections
- approximate controllability of the original system in modulus
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- controllability of the Galerkyn approximations
- convexification procedure: high-order Galerkyn approximations’ dynamics can mimic low order ones in projections
- approximate controllability \textit{in modulus} in observed projections
- approximate controllability of the original system in modulus
- phases adjustment
Exemple: rotating bipolar molecule

\[ i \frac{\partial \psi(\theta, t)}{\partial t} = \left( -\frac{\partial^2}{\partial \theta^2} + u(t) \cos(\theta) \right) \psi(\theta, t), \quad \theta \in S^1 \]

- \( \theta \) rotational degree of freedom of a bipolar rigid molecule confined to a plane
- controlled fields pointing in the direction (0, 1)
- \( u(t) \in [0, \delta] \)
- the spaces of even and odd wavefunctions are invariant (no transfer of probability between them)
Lemma

The system is approximately controllable.

\( H_e \): even wavefunctions; \( H_o \): odd wavefunctions
Complete orthonormal systems for \( H_e \) and \( H_o \) of eigenfunctions of \( A \) are given by \( \{\cos(k(\cdot))/\sqrt{\pi}\}_{k=0}^{\infty} \) and \( \{\sin(k(\cdot))/\sqrt{\pi}\}_{k=1}^{\infty} \), respectively.

The sufficient conditions for controllability are easily tested:

\[
A = \text{diag}(k^2/\pi)
\]

\[
B = \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 & \cdots \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & 0 & \cdots \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

\[
S = \{(j, k) \mid |j - k| = 1\}
\]