Reconstruction of a perfectly conducting obstacle coated with a thin dielectric layer

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The scattering problem for coated obstacles in the harmonic regime

\[ \text{div}(\epsilon^{-1} \nabla u_\delta) + \mu k^2 u_\delta = 0 \]

\[ \frac{\partial u_\delta}{\partial \nu_\delta} = 0 \]

\[ \Delta u_\delta + k^2 u_\delta = 0 \]

\[ u_\delta = u^s_\delta + u^i \]

\[ D = \text{Gray + Red} \]

\[ \lim_{R \to \infty} \int_{|x| = R} |\partial_r u^s_\delta - i ku^s_\delta|^2 ds = 0 \]
The inverse scattering problem for coated obstacles

$$\text{div}(\epsilon^{-1} \nabla u_\delta) + \mu k^2 u_\delta = 0$$

$$\frac{\partial u_\delta}{\partial \nu_\delta} = 0$$

$$\Delta u_\delta + k^2 u_\delta = 0$$

$$u_\delta = u_\delta^s + u^i$$

$$\lim_{R \to \infty} \int_{|x|=R} |\partial_r u_\delta^s - ik u_\delta^s|^2 \, ds = 0.$$ 

Qualitative techniques

+ Fast
- Need a lot of data
- Not very accurate

Quantitative techniques

+ Provide accurate reconstruction
- Need a priori information
- Computational cost

Design an optimization procedure to find the shape and the thickness from scattered field data.
Outline

1. Formulation of the inverse problem
   - An optimization point of view
   - A GIBC approximate model

2. Derivatives of the cost function

3. Numerical experiments
The inverse problem

The far field map

For $u^i(x, \hat{\theta}) = e^{ik\hat{\theta} \cdot x}$ define

$T^\text{coat} : (\epsilon, \mu, \delta, \Gamma, \hat{\theta}) \mapsto u^\infty(\hat{x}, \hat{\theta})$

where $u^\infty$ associated with $u^s$ is defined in dimension $d$ by

$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left( u^\infty(\hat{x}) + O\left(\frac{1}{r}\right) \right) \quad r \to +\infty.$

The inverse problem

Given $N$ far–fields $(u^\infty_{\text{coat}}(\cdot, \hat{\theta}_j))_{j=1,\ldots,N}$, retrieve the geometry $\Gamma$ and the properties of the layer.
The inverse problem

The far field map

For \( u^i(x, \hat{\theta}) = e^{ik\hat{\theta} \cdot x} \) define

\[
T^{\text{coat}} : (\epsilon, \mu, \delta, \Gamma, \hat{\theta}) \mapsto u^\infty(\hat{x}, \hat{\theta})
\]

where \( u^\infty \) associated with \( u^s \) is defined in dimension \( d \) by

\[
u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left( u^\infty(\hat{x}) + O\left(\frac{1}{r}\right)\right) \quad r \to +\infty.
\]

The inverse problem

Given \( N \) far–fields \( (u^\infty_{\text{coat}}(\cdot, \hat{\theta}_j))_{j=1,\ldots,N} \), retrieve the geometry \( \Gamma \) and the properties of the layer.

Minimize

\[
F^{\text{coat}}(\epsilon, \mu, \delta, \Gamma) := \frac{1}{2} \sum_{j=1}^{N} \| T^{\text{coat}}(\epsilon, \mu, \delta, \Gamma, \hat{\theta}_j) - u^\infty_{\text{coat}}(\cdot, \hat{\theta}_j) \|_{L^2(S_j)}^2
\]

\( T^{\text{coat}} \) is computationally expensive to evaluate!
The inverse problem

The far field map

For \( u^i(x, \hat{\theta}) = e^{ik\hat{\theta} \cdot x} \) define

\[
T^{\text{coat}} : (\epsilon, \mu, \delta, \Gamma, \hat{\theta}) \mapsto u^\infty(\hat{x}, \hat{\theta})
\]

The inverse problem

Given \( N \) far–fields \((u_{\text{coat}}^\infty(\cdot, \hat{\theta}_j))_{j=1,\ldots,N}\), retrieve the geometry \( \Gamma \) and the properties of the layer.

Minimize

\[
F^{\text{coat}}(\epsilon, \mu, \delta, \Gamma) := \frac{1}{2} \sum_{j=1}^{N} \|T^{\text{coat}}(\epsilon, \mu, \delta, \Gamma, \hat{\theta}_j) - u_{\text{coat}}^\infty(\cdot, \hat{\theta}_j)\|_{L^2(S_j)}^2
\]

\( T^{\text{coat}} \) is computationally expensive to evaluate!

Minimize

\[
F^{\text{equi}}(\epsilon, \mu, \delta, \Gamma) := \frac{1}{2} \sum_{j=1}^{N} \|T^{\text{equi}}(\epsilon, \mu, \delta, \Gamma, \hat{\theta}_j) - u_{\text{coat}}^\infty(\cdot, \hat{\theta}_j)\|_{L^2(S_j)}^2
\]
An approximate GIBC model

\[ \|u_\delta - u_m\| \leq C\delta^{m+1/2} \]

\[ \text{Exact model} \]

\[ \text{Equivalent model of order } m \]

In dimension 2: (Aslanyürecek, Haddar, Şahintürk [11])

\[ Z_1 = \frac{\partial}{\partial s} \delta \epsilon^{-1} \frac{\partial}{\partial s} + \delta k^2 \mu \]

\[ Z_2 = \frac{\partial}{\partial s} \left( \delta - \frac{\delta^2 c}{2} \right) \epsilon^{-1} \frac{\partial}{\partial s} + \left( \delta + \frac{\delta^2 c}{2} \right) k^2 \mu. \]
The GIBC forward problem

Find \( u = u^s + u^i \) such that

\[
\begin{align*}
    u^s \in \left\{ v \in \mathcal{D}'(\Omega_{\text{ext}}), \right. \\
    \left. \varphi v \in H^1(\Omega_{\text{ext}}) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d); \quad v|_{\Gamma} \in H^1(\Gamma) \right\}
\end{align*}
\]

and

\[
\begin{cases}
    \Delta u + k^2 u = 0 & \text{in } \Omega_{\text{ext}} \\
    \frac{\partial u}{\partial \nu} + \text{div}_\Gamma (\eta \nabla_\Gamma u) + \lambda u = 0 & \text{on } \Gamma \\
    \lim_{R \to \infty} \int_{|x| = R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0.
\end{cases}
\]

\( u \) exists and is unique if

1. \( \Im m(\lambda) \geq 0, \ \Im m(\eta) \leq 0 \) a.e. on \( \Gamma \) \quad (\text{physical assumption})
2. \( \Re e(\eta) \geq c \) \quad a.e. on \( \Gamma \) for \( c > 0 \).
Reformulation of the inverse problem

The far field map for the GIBC model

For \( u^i(x, \hat{\theta}) = e^{ik\hat{\theta} \cdot x} \) define

\[
T^{GIBC} : (\lambda, \eta, \Gamma, \hat{\theta}) \mapsto u^\infty(\hat{x}, \hat{\theta}).
\]

For thin layer we have:

\[
T^{GIBC}(\delta k^2 \mu, \delta \epsilon^{-1}, \Gamma, \hat{\theta}_j) = T^{coat}(\epsilon, \mu, \delta, \Gamma, \hat{\theta}_j) + O(\delta^{3/2})
\]

New functional to minimize

Given \( N \) far–fields \( (u^\infty_{coat}(\cdot, \hat{\theta}_j))_{j=1,\ldots,N} \) corresponding to the coated obstacle, minimize

\[
F^{GIBC}(\lambda, \eta, \Gamma) := \frac{1}{2} \sum_{j=1}^{N} \| T^{GIBC}(\lambda, \eta, \Gamma, \hat{\theta}_j) - u^\infty_{coat}(\cdot, \hat{\theta}_j) \|_{L^2(S_j)}^2
\]
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Difficulty related to the shape derivative

GIBC model: \[ \frac{\partial u}{\partial \nu} + \text{div}_\Gamma(\eta \nabla_\Gamma u) + \lambda u = 0 \quad \text{on } \Gamma \]

\[ F^{GIBC}(\lambda, \eta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| T^{GIBC}(\lambda, \eta, \Gamma, \hat{\theta}_j) - u^\infty_{\text{coat}}(\cdot, \hat{\theta}_j) \|_{L^2(S_j)}^2 \]

For minimizing \( F^{GIBC} \) we use a steepest descent method:

- we need partial derivatives of the far-field with respect to \( \lambda \) and \( \eta \) (quite standard),
- we need an appropriate derivative w.r.t. the obstacle.

Difficulty: the unknown impedances are supported by \( \Gamma \).
Derivative of the cost function with respect to the obstacle

\[ h \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \text{ is “small”} \]

\[ f_h := \text{Id} + h \]

\[ \Gamma_h := f_h(\Gamma) \]

\[ \lambda \text{ and } \eta \text{ being constant} \]

We define the derivative \( v_h \) of the scattered field with respect to the geometry at point \((\lambda, \eta, \Gamma)\) by

\[ u^s(\lambda, \eta, \Gamma_h) - u^s(\lambda, \eta, \Gamma) = v_h + o(||h||) \]

where \( h \mapsto v_h \) is linear.
Derivative of the cost function with respect to the obstacle

\( h \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \) is “small”

\[
\begin{align*}
  f_h &:= \text{Id} + h \\
  \Gamma_h &:= f_h(\Gamma) \\
  \lambda_h &:= \lambda \circ f_h^{-1}, \quad \eta_h := \eta \circ f_h^{-1}
\end{align*}
\]

We define the derivative \( \nu_h \) of the scattered field with respect to the geometry at point \((\lambda, \eta, \Gamma)\) by

\[
\begin{align*}
  u^s(\lambda_h, \eta_h, \Gamma_h) - u^s(\lambda, \eta, \Gamma) = \nu_h + o(||h||)
\end{align*}
\]

where \( h \mapsto \nu_h \) is linear.
Derivative of the cost function with respect to the obstacle

\[ h \in C^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \text{ is “small”} \]

\[ f_h := \text{Id} + h \]

\[ \Gamma_h := f_h(\Gamma) \]

\[ \lambda_h := \lambda \circ f_h^{-1}, \quad \eta_h := \eta \circ f_h^{-1} \]

We define the derivative \( v_h \) of the scattered field with respect to the geometry at point \((\lambda, \eta, \Gamma)\) by

\[ u^s(\lambda_h, \eta_h, \Gamma_h) - u^s(\lambda, \eta, \Gamma) = v_h + o(||h||) \]

where \( h \mapsto v_h \) is linear.

One may find \( f_h \) such that \( \Gamma = f_h(\Gamma) \) and

\[ F'_{\lambda,\eta}(\Gamma) \cdot h \neq 0. \]

\( F'_{\lambda,\eta}(\Gamma) \) does not satisfy the classical shape derivative’s properties!
Derivative of the scattered field with respect to the obstacle

Let \((\lambda, \eta, \Gamma)\) be given and analytic, for some small \(h \in C^{1,\infty}\) define

\[ \Gamma_h = f_h(\Gamma), \quad \lambda_h := \lambda \circ f_h^{-1} \quad \text{and} \quad \eta_h := \eta \circ f_h^{-1}. \]

Let \(u^s_h [u^s]\) be scattered field associated with \((\lambda_h, \eta_h, \Gamma_h) [((\lambda, \eta, \Gamma))].\)

\[
\begin{align*}
    u^s_h(x) - u^s(x) &= v_h(x) + o(||h||),
\end{align*}
\]

where \(v_h(x)\) is the solution of the scattering problem with

\[
\begin{align*}
    \frac{\partial v_h}{\partial \nu} + Z v_h &= B_h u \quad \text{on} \quad \Gamma \\
    B_h u &= (h \cdot \nu)(k^2 - 2H \lambda)u + \text{div}_\Gamma ((Id + 2\eta(R - H Id))(h \cdot \nu) \nabla_{\Gamma} u) \\
    &\quad + (\nabla_{\Gamma} \lambda \cdot h)u + \text{div}_\Gamma ((\nabla_{\Gamma} \eta \cdot h) \nabla_{\Gamma} u) \\
    &\quad + Z ((h \cdot \nu)Z u),
\end{align*}
\]

with \(2H := \text{div}_\Gamma \nu, \; R := \nabla_{\Gamma} \nu\) and \(Z \cdot = \text{div}_\Gamma (\eta \nabla_{\Gamma} \cdot) + \lambda.\)
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**Numerical algorithm**

\[
F^{\text{GIBC}}(\lambda, \eta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| T^{\text{GIBC}}(\lambda, \eta, \Gamma, \hat{\theta}_j) - u_{\text{obs}}^\infty(\cdot, \hat{\theta}_j) \|_{L^2(S_j)}^2
\]

**Numerical procedure:**

- update alternatively \( \lambda, \eta \) and \( \Gamma \) with a direction given by the partial derivative of the cost function,

\[
\lambda^{n+1} = \lambda^n - \alpha_n F'_{\eta,\Gamma}(\lambda^n)
\]
The regularization procedure

\[ F^{\text{GIBC}}(\lambda, \eta, \Gamma) = \frac{1}{2} \sum_{j=1}^{I} \|T^{\text{GIBC}}(\lambda, \eta, \Gamma, \hat{\theta}_j) - u_{\text{obs}}^{\infty}(\cdot, \hat{\theta}_j)\|_{L^2(S_j)}^2 \]

We regularize the gradient, NOT the cost function, using a \( H^1(\Gamma) \) regularization.

- Descent direction for \( \lambda \): find \( \delta \lambda \) that solves for every \( \phi \) in some finite dimensional space:

\[
\beta \lambda \int_{\Gamma} \nabla_{\Gamma}(\delta \lambda) \cdot \nabla_{\Gamma} \phi \, ds + \int_{\Gamma} \delta \lambda \phi \, ds = -\alpha \lambda \, F'_{\eta, \Gamma}(\lambda) \cdot \phi
\]

where \( \beta \lambda \) is the \textit{regularization coefficient} and \( \alpha \lambda \) is the descent coefficient.

- Do the same for \( \delta \eta \) and \( \delta \Gamma \).
Numerical reconstruction

Finite elements method and remeshing procedure

using FreeFem++

Reconstruction of the geometry with 2 incident waves and 1% noise on the far-field, $\lambda = ik/2$ and $\eta = 2/k$ being known.

The synthetic data come from the GIBC model
Application to the reconstruction of a coated obstacle

**Exact model**

\[
\text{div}(\epsilon^{-1} \nabla u_{\delta}) + \mu k^2 u_{\delta} = 0
\]

\[
\Delta u_{\delta} + k^2 u_{\delta} = 0
\]

**Equivalent model of order 1**

\[
\frac{\partial u_1}{\partial \nu} + \text{div}(\delta \epsilon^{-1} \nabla u_1) + \delta \mu k^2 u_1 = 0
\]

\[
\Delta u_1 + k^2 u_1 = 0
\]

Reconstruction of an obstacle using the generalized impedance boundary condition model of order 1 minimizing

\[
F^{GIBC}(\mu, \delta, \Gamma) := \frac{1}{2} \sum_{j=1}^{I} \| T^{GIBC}(\mu, \delta, \Gamma, \hat{\theta}_j) - u^\infty_{\text{coat}}(\cdot, \hat{\theta}_j) \|_{L^2(S_j)}^2
\]

with \( \epsilon = 0.1 \) known.
Application to the reconstruction of a coated obstacle

Numerical results

Artificial data created with
- $\epsilon = 0.1$ is known,
- $\delta = 0.04l(1 - 0.4 \sin(\theta))$ is unknown; $l$ being the wavelength,
- $\mu = 2.5$ is unknown.

Reconstructed $\mu$: 2.3.

Fails with a classical impedance boundary condition model!
Conclusion

- The quantitative reconstruction of coated obstacles can be quite fast and accurate.
- The optimization methods also provide the thickness of the coating as well as its physical parameters.

- Extension to the 3D Maxwell equations.
- Can be applied for any complex structure that can be approximated by a GIBC.