Asymptotic of the velocity of a dilute suspension of droplets with interfacial tension

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We consider a small deformable droplet immersed in an incompressible Newtonian fluid.

Droplet can be seen as an inhomogeneity of small size $\varepsilon$ in the Newtonian fluid.

Can we derive an asymptotic expansion of the velocity $u_\varepsilon$ as the droplet diameter $\varepsilon$ tends to 0?

Asymptotic expansion is given in terms of a polarization tensor $P$.

What is the expression of $P$? Can we determine an explicit expression for particular shapes of the droplet?
Outline

1. The mathematical model of a suspension of droplet
2. Polarization tensor and asymptotic of the velocity
3. Polarization tensor of an ellipse
4. Conclusion and perspectives
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Homogeneous matrix fluid

The matrix fluid is a stationary Stokes flow in a bounded $C^2$-domain $\Omega$ of $\mathbb{R}^n$ ($n = 2, 3$). Then, the velocity $u$ and pressure $p$ of the fluid satisfy:

\[
\begin{cases}
-\text{Div}(2\mu_0 e(u)) + \nabla p = 0 & \text{in } \Omega, \\
\text{div}(u) = 0 & \text{in } \Omega, \\
\text{div}(u) = 0 & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega.
\end{cases}
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where $g \in C^\infty(\partial \Omega)^n$ satisfies the compatibility condition

\[
\int_{\partial \Omega} g \cdot \nu_x \, d\sigma_x = 0.
\]

The regularity assumption on $\Omega$ is necessary in order to have the regularity of $u$ in $\Omega$, away from the boundary $\partial \Omega$. 

Droplet immersed in the matrix fluid

The droplet lies in a bounded $C^2$-domain $\omega_\varepsilon \subset \Omega$ of small size $\varepsilon > 0$ given by

$$\omega_\varepsilon := z_0 + \varepsilon \omega,$$

where $z_0 \in \Omega$ satisfies $\text{dist}(z_0, \partial \Omega) > 0$. $\omega$ contains the origin and represents the droplet shape, while $z_0$ is the position of the droplet.
**Droplet immersed in the matrix fluid**

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$\omega_\varepsilon$ contains a Stokes flow with viscosity $\mu_1 \neq \mu_0$. We define the viscosity $\mu_\varepsilon$ in $\Omega$ by

$$\mu_\varepsilon(x) := \begin{cases} 
\mu_0 & \text{if } x \in \Omega \setminus \overline{\omega_\varepsilon}, \\
\mu_1 & \text{if } x \in \omega_\varepsilon.
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Denote by $\kappa_\varepsilon \in L^\infty(\partial \omega_\varepsilon)$ the mean curvature of $\partial \omega_\varepsilon$. The surface tension $T_\varepsilon$ on the boundary $\partial \omega_\varepsilon$ is given by

$$T_\varepsilon := \lambda_\varepsilon \kappa_\varepsilon \nu_x,$$

where $\lambda_\varepsilon$ is the constant surface tension coefficient associated with the inclusion $\omega_\varepsilon$ and $\nu_x$ is the outward unit normal to $\partial \omega_\varepsilon$. 
Let $\chi_{\partial \omega_\varepsilon}$ be the characteristic function of the boundary $\partial \omega_\varepsilon$. Then, the velocity $u_\varepsilon$ and the pressure $p_\varepsilon$ of the suspension of droplet are the solutions of the following problem:

$$\begin{cases}
-\text{Div}(2\mu_\varepsilon e(u_\varepsilon)) + \nabla p_\varepsilon = \lambda_\varepsilon \kappa_\varepsilon \nu_x \chi_{\partial \omega_\varepsilon} & \text{in } \Omega, \\
\text{div}(u_\varepsilon) = 0 & \text{in } \Omega, \\
u_\varepsilon = g & \text{on } \partial \Omega,
\end{cases}$$

(2)
Let $\chi_{\partial \omega_{\varepsilon}}$ be the characteristic function of the boundary $\partial \omega_{\varepsilon}$. Then, the velocity $u_{\varepsilon}$ and the pressure $p_{\varepsilon}$ of the suspension of droplet are the solutions of the following problem:

$$
\begin{aligned}
-\text{Div}(2 \mu_{\varepsilon} \varepsilon(u_{\varepsilon})) + \nabla p_{\varepsilon} &= \lambda_{\varepsilon} \kappa_{\varepsilon} \nu \chi_{\partial \omega_{\varepsilon}} & \text{in } \Omega, \\
\text{div}(u_{\varepsilon}) &= 0 & \text{in } \Omega, \\
\quad u_{\varepsilon} &= g & \text{on } \partial \Omega,
\end{aligned}
$$

(2)

- We are interested in deriving an asymptotic expansion of the velocity $u_{\varepsilon}$ as the droplet diameter $\varepsilon$ tends to 0.

- For this purpose, we introduce the Green tensor $G : \Omega \rightarrow \mathbb{R}^{n \times n}$ of the matrix fluid, i.e. the fundamental solution of the Stokes equation with viscosity $\mu_0$ in $\Omega$. 
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Denote by \( G_i \) the \( i \)th row of \( G \), \( i = 1, \ldots, n \).

**Theorem**

For any \( z \in \Omega \) at a distance \( d > 0 \) away from \( \omega_\varepsilon \) and for \( i \in \{1, \ldots, n\} \), we have

\[
(u_\varepsilon - u)(z)_i = \varepsilon^n e_x(G_i)(z, z_0) : V e_x(u)(z_0) \\
+ \varepsilon^n e_x(G_i)(z, z_0) : K + O(\varepsilon^{n+\frac{1}{2}}),
\]

(3)

where \( V \) is a fourth order symmetric tensor and \( K \) is a symmetric matrix. The term \( O(\varepsilon^{n+\frac{1}{2}}) \) is uniformly bounded by \( c \varepsilon^{n+\frac{1}{2}} \) where the constant \( c \) depends on \( d \), \( \mu_0 \) and \( \mu_1 \).
Asymptotic expansion

Denote by $G_i$ the $ith$ row of $G$, $i = 1, \ldots, n$.

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where $V$ is a fourth order symmetric tensor and $K$ is a symmetric matrix. The term $O(\varepsilon^{n+\frac{1}{2}})$ is uniformly bounded by $c \varepsilon^{n+\frac{1}{2}}$ where the constant $c$ depends on $d, \mu_0$ and $\mu_1$.

→ The tensor $V$ is the viscous moment tensor introduced by Ammari, Garapon, Kang & Lee (2008) for a zero surface tension.

→ Our contribution in this work is the consideration of surface tension that appears through the tensor $K$ called curvature moment tensor.
Definition of the viscous moment tensor \( V \)

Let \( \mu \) be the rescaled viscosity, \( i.e. \)

\[
\mu(y) := \begin{cases} 
\mu_0 & \text{if } y \in \mathbb{R}^n \setminus \bar{\omega}, \\
\mu_1 & \text{if } y \in \omega.
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The symmetric fourth order tensor $V$ has for coefficients

$$V_{ijkl} := 2(\mu_0 - \mu_1) \left( \frac{|\omega|}{2} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \int_{\omega} e(\varphi^{kl})_{ij} \, dy \right), \quad (4)$$

where $(\varphi^{kl}, s^{kl}) \in H^1_{\text{loc}}(\mathbb{R}^n)^n \times L^2_{\text{loc}}(\mathbb{R}^n)$ is the solution to
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$$
\begin{cases}
-\text{Div}(2\mu \, e(\varphi^{kl})) + \nabla s^{kl} \\
= (\mu_0 - \mu_1) \left( (e^l \nu_k + e^k \nu_l) + \frac{2}{n} \nu_y \delta_{kl} \right) \chi_{\partial \omega} & \text{in } \mathbb{R}^n, \\
\text{div}(\varphi^{kl}) = 0 & \text{in } \mathbb{R}^n, \\
\varphi^{kl}(y) = O(|y|^{-1}) & \text{when } |y| \to \infty, \\
s^{kl}(y) = O(|y|^{-2}) & \text{when } |y| \to \infty.
\end{cases}
$$
Definition of the curvature moment tensor $K$

The curvature moment tensor $K$ is the symmetric matrix with zero trace of coefficients

$$K_{ij} := 2 \tilde{\lambda} (\mu_0 - \mu_1) \int_\omega e(\varphi)_{ij} \, dy - \tilde{\lambda} \int_{\partial \omega} \left( \nu_i \nu_j - \frac{\delta_{ij}}{n} \right) \, d\sigma_y, \quad (5)$$

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\[
\begin{cases}
-\text{Div}(2\mu \, e(\varphi)) + \nabla s = \kappa \, \nu_y \, \chi_{\partial\omega} & \text{in } \mathbb{R}^n, \\
\text{div}(\varphi) = 0 & \text{in } \mathbb{R}^n, \\
\varphi(y) = O(|y|^{-1}) & \text{when } |y| \to \infty, \\
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and $\tilde{\lambda} = \frac{\lambda}{\delta}$ is a constant with $\lambda$ the constant surface tension associated with $\partial \omega$ and $\delta$ the Tolman length.
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and $\tilde{\lambda} = \frac{\lambda}{\delta}$ is a constant with $\lambda$ the constant surface tension associated with $\partial\omega$ and $\delta$ the Tolman length.

Simple computations show that if $\kappa$ is constant (for example if $\omega$ is a ball) then $K = 0$. 
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Tensor $\mathbf{V}$ for an ellipse

Ammari, Garapon, Kang & Lee have obtained an explicit formula of the viscous moment tensor $\mathbf{V}$ for an ellipse.

If $\omega$ is an ellipse given by

$$x_1^2/a^2 + x_2^2/b^2 = 1$$

where $a \geq b > 0$. (7)

Then, the non zero coefficients of $\mathbf{V}$ are

$$
\begin{align*}
V_{1111} &= V_{2222} = -V_{1122} = -V_{2211} = 2\mu_0 (\mu_1 - \mu_0) (a + b)\frac{|\omega|}{(a + b)^2} \\
V_{1212} = V_{2112} = V_{1221} = V_{2121} &= 2\mu_0 (\mu_1 - \mu_0) (a + b)\frac{|\omega|}{(a + b)^2} + (\mu_1 - \mu_0) (a - b)^2
\end{align*}
$$

In particular if $\omega$ is a disk, we have

$$\mathbf{V} = 4 |\omega| \mu_0 \mu_1 - \mu_0 \mu_1 + \mu_0 (I - 1/2 I_2 \otimes I_2)$$
Tensor $V$ for an ellipse

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If $\omega$ is an ellipse given by

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \quad \text{where} \quad a \geq b > 0. \quad (7)$$

Then, the non zero coefficients of $V$ are

$$V_{1111} = V_{2222} = -V_{1122} = -V_{2211}$$

$$= \frac{2\mu_0(\mu_1 - \mu_0)(a + b)^2|\omega|}{(\mu_0 + \mu_1)(a + b)^2 + (\mu_0 - \mu_1)(a - b)^2},$$

$$V_{1212} = V_{2112} = V_{1221} = V_{2121}$$

$$= \frac{2\mu_0(\mu_1 - \mu_0)(a + b)^2|\omega|}{(\mu_0 + \mu_1)(a + b)^2 + (\mu_1 - \mu_0)(a - b)^2}.$$
Tensor $K$ for an ellipse

**Theorem**

If $\omega$ is an ellipse given by (7), the curvature moment tensor is

$$K(a, b) = 2\tilde{\lambda} \left\{ \frac{\mu_1(a + b)^2 + (\mu_0 - \mu_1)(a - b)^2}{(\mu_0 + \mu_1)(a + b)^2 + (\mu_0 - \mu_1)(a - b)^2} \chi(a, b) \right\} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),$$

where $\chi(a, b) \geq 0$ is defined by

$$\chi(a, b) := \int_0^{2\pi} \frac{a^2 \sin^2 \theta - b^2 \cos^2 \theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \, d\theta.$$
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**Theorem**

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$$K(a, b) = 2\tilde{\lambda} \frac{\mu_1 (a + b)^2 + (\mu_0 - \mu_1)(a - b)^2}{(\mu_0 + \mu_1)(a + b)^2 + (\mu_0 - \mu_1)(a - b)^2} \chi(a, b) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(8)

where $\chi(a, b) \geq 0$ is defined by

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(9)

**Remark**

- If $\omega$ is a disk, we obtain $K = 0$.
- If we fix $|\omega|$, we can written (8) in function of the eccentricity $0 \leq e \leq 1$ of the ellipse.
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Results obtained:

- Polarization tensor \( K \) which depends on the surface tension,
- Explicit formulation of \( K \) for an ellipse,
- Linear system satisfies by \( K \) for an ellipsoid.
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Perspectives:

- Dynamical case (non stationary Stokes + droplet $\omega(t)$),
- Describe the dynamics of the droplet shape,
- Consider vesicles/red blood cells (coupling Navier-Stokes, elastic membrane + surface forces).