Signal to noise ratio estimation in passive correlation based imaging

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Problèmes Inverses, Contrôle et Optimisation de Formes
École Polytechnique, Palaiseau, 2-4 April 2012
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Idea: use cross-correlations between pairs of sensors (receivers) to retrieve information about the Green’s function in the background medium. This can be achieved either with equi-distribution of sources or through multiple scattering.

Our goal

- Our aim is to use these cross-correlations in order to image reflectors (embedded in clutter).

Application: Structural Health Monitoring


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Application : Structural Health Monitoring


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Mathematical model
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- We consider a domain $\Omega$ containing a reflector $\mathcal{O}$
- We consider $u(t, x)$ solution of the time-dependent wave equation

$$\begin{cases}
\frac{\partial^2 u}{\partial t^2}(t, x) - c_0^2 \Delta u = n(t, x) \quad \text{in } \Omega \setminus \mathcal{O} \\
u(t, x) = 0 \quad \text{on } \partial\mathcal{O} \\
+ \text{ PML (to model free-space problem)}
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- $c_0$ is homogeneous propagation speed.
- $n(t, x)$ models noise sources.
Noise source term

- \( n(t, x) \) is a zero mean stationary (in time) random process:

\[
E \{ n(t, x) \} = 0
\]
A noise source term

- $n(t, x)$ is a zero mean stationary (in time) random process:

$$E\{n(t, x)\} = 0$$

- $n(t, x)$ satisfies the following cross-correlation relation

$$E\{n(t_1, x_1)n(t_2, x_2)\} = \mathbb{F}(t_2 - t_1)K(x_1)\delta(x_1 - x_2)$$

$\mathbb{F}(\cdot)$ is a real-valued even function, has its maximum on $t = 0$ and has a positive Fourier transform $K(\cdot)$ characterize the spatial support of noise sources.
Noise source term

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  E\{n(t_1, x_1)n(t_2, x_2)\} = F(t_2 - t_1)K(x_1)\delta(x_1 - x_2)
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- \( F(t) \) is a real-valued even function, has its maximum on \( t = 0 \) and has a positive Fourier transform
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- \( K(x) \) characterize the spatial support of noise sources.
Cross correlations

- Assume that we know $u(t, x_1)$ and $u(t, x_2)$ the time-dependant wave fields recorded by two sensors at $x_1$ and $x_2$ on a time interval $[0, T]$.
Cross correlations

- assume that we know $u(t, x_1)$ and $u(t, x_2)$ the time-dependant wave fields recorded by two sensors at $x_1$ and $x_2$ on a time interval $[0, T]$

- their cross-correlation function over the time interval $[0, T]$, and with time lag $\tau$ is given by

$$C_T(\tau, x_1, x_2) = \frac{1}{T} \int u(t, x_1)u(t + \tau, x_2)dt$$
The expectation of $C_T$ (with respect to the distribution of the sources) is independent of $T$:

$$E \{ C_T(\tau, x_1, x_2) \} = C^{(1)}(\tau, x_1, x_2)$$

with

$$C^{(1)}(\tau, x_1, x_2) = \frac{1}{2\pi} \int \hat{D}(\omega, x_1, x_2) \hat{F}(\omega) e^{-i\omega \tau} d\omega$$

$$\hat{D}(\omega, x_1, x_2) = \int \overline{\hat{G}(\omega, x_1, y)} \hat{G}(\omega, x_2, y) K(y) dy$$
Properties

- The expectation of $\mathcal{C}_T$ (with respect to the distribution of the sources) is independent of $T$:

$$\mathbb{E}\{\mathcal{C}_T(\tau, x_1, x_2)\} = \mathcal{C}^{(1)}(\tau, x_1, x_2)$$

with

$$\mathcal{C}^{(1)}(\tau, x_1, x_2) = \frac{1}{2\pi} \int \hat{D}(\omega, x_1, x_2) \hat{F}(\omega) e^{-i\omega \tau} d\omega$$

$$\hat{D}(\omega, x_1, x_2) = \int \hat{G}(\omega, x_1, y) \hat{G}(\omega, x_2, y) K(y) dy$$

- The empirical cross correlation $\mathcal{C}_T$ is a statistical stable self-averaging quantity, i.e.:

$$\mathcal{C}_T(\tau, x_1, x_2) \xrightarrow{T \to \infty} \mathcal{C}^{(1)}(\tau, x_1, x_2)$$
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Assumptions

- noise sources are spatially localised (gray region)
- passive sensors \((x_j)_{1 \leq j \leq J}\) are located between the sources and the reflector
Stationary phase analysis

Stationary phase analysis done by J. Garnier and G. Papanicolaou shows that the cross correlation \( C^{(1)}(\tau, x_1, x_2) \) between two sensors \( x_1 \) and \( x_2 \) has two peaks

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- One peak (direct arrival) at the travel time between $x_1$ and $x_2$
  \[ |x_1 - x_2| \]
  \[ \frac{c_0}{c_0} \]

- One peak (reflected arrival) at the sum of the travel times between $x_i$ and the reflector
  \[ |x_1 - z_O| \]
  \[ c_0 \]
  \[ |z_O - x_2| \]
  \[ c_0 \]

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- one peak (direct arrival) at the travel time between \( x_1 \) and \( x_2 \)
  \[
  \frac{|x_1 - x_2|}{c_0}
  \]

- one peak (reflected arrival) at the sum of the travel times between \( x_i \) and the reflector
  \[
  \frac{|x_1 - z_O|}{c_0} + \frac{|z_O - x_2|}{c_0}
  \]

To keep only the interesting peak that concerns the reflector, the image at a search point $z$ is computed using the following daylight imaging functional:

$$I^D(z) = \sum_{j,l=1}^{J} C^{(1),\text{sym}}_{\text{coda}}(\tau(z, x_l) + \tau(z, x_j), x_j, x_l),$$

**Proposition** If we consider that the reflector is far enough from the receivers and $T$ is large enough, then we can approximate $I^D(z)$ by

$$I^D(z) \approx \sum_{j,l=1}^{J} C^{(1),\text{sym}}_{\text{coda}}(\tau(z, x_l) + \tau(z, x_j), x_j, x_l).$$
Daylight imaging functional

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- $C_{T,\text{coda}}^{\text{sym}}$ is defined by

$$C_{coda}^{(1),{\text{sym}}} (t, x_j, x_l) = \left( C^{(1)}(t, x_j, x_l) + C^{(1)}(-t, x_j, x_l) \right) \mathbf{1}_{\tau(x_j, x_l), +\infty}(t).$$
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Proposition

If we consider that the reflector is far enough from the receivers and $T$ is large enough, then we can approximate $I^D(z)$ by

$$I^D(z) \simeq 2 \sum_{j,l=1}^{J} C_T(\tau(z, x_l) + \tau(z, x_j), x_j, x_l),$$
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Simulation setup

- wave equation on the rectangle $[0, 50\lambda] \times [-15\lambda, 15\lambda]$, with a reflector located on $[44\lambda, 46\lambda] \times [-\lambda, \lambda]$, 
- random distribution of sources has support on the rectangle $\Omega_S = [0, 4\lambda] \times [-15\lambda, 15\lambda]$, 
- we record the solution $u$ of the wave equation at $J$ receivers located at $x_j = (5\lambda, (j - (J + 1)/2)\lambda/2)$, for $1 \leq j \leq J$, 
- $c_0 = 3\text{km s}^{-1}$, 

Figure: Geometry of the passive sensor imaging problem for a daylight illumination.
Dependence in space: \( K(x) \) is given by

\[
K(x) = \frac{1}{|\Omega_S|} \mathbb{1}_{\Omega_S}(x)
\]
**Source term**

- Dependence in space: \( K(x) \) is given by
  \[
  K(x) = \frac{1}{|\Omega_S|} \mathbb{1}_{\Omega_S}(x)
  \]

- Dependence in time: \( F(t) \) is given by
  \[
  F(t) = \frac{1}{3T} \int F(\tau)F(t + \tau) d\tau \\
  F(t) = \frac{\sin(B(t - T/2))}{B(t - T/2)} \cos(2\pi f(t - T/2)) \exp\left(-\frac{(t - T/2)^2}{2ct^2}\right)
  \]

- \( B = 0.3 \text{ Hz} \) is the bandwidth
- \( f = 0.3 \text{ Hz} \) is the central frequency
- \( c_t = 2.5 \text{ s} \) is the correlation time
Computation

- Space discretization is done by 8-th order mixed spectral finite elements with Gauss-Lobatto points
- Time discretization is done by 4-th order Runge-Kutta.
- We implement the source term by

\[ n(t, x) = \frac{1}{\sqrt{N_s}} \sum_{s=1}^{N_s} \delta(x - x_s) \hat{F}^{-1} (r_s \mathcal{F}(F)) (t) \]

\( r_s(\omega) \) is a random distribution satisfying
\( r_s(-\omega) = r_s(\omega) \) and following expectations:
\[ E \{ r_s(\omega_1) r_s(\omega_2) \} = \frac{1}{3} \delta(\omega_1 + \omega_2) \]
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\[ \mathbb{E}\{r_s(\omega)\} = 0 \] and \[ \mathbb{E}\{r_s(\omega_1)r_s(\omega_2)\} = \frac{1}{3} \delta(\omega_1 + \omega_2) \]
Daylight imaging functional

Figure: Daylight imaging functional for the homogeneous medium. $J = 21$
Figure: Daylight imaging functional for the homogeneous medium. $J = 31$
Figure: Daylight imaging functional for the homogeneous medium. $J = 41$
Figure: Daylight imaging functional for the homogeneous medium. $J = 51$
A good question that naturally arise is “Under what conditions do we obtain such a good image”?
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A partial answer to this question is that the signal to noise ratio (SNR) of the image increases with the number of the receivers, where

\[
\text{SNR} = \frac{|\mathcal{I}^D|({z^*})}{\max_{z \neq z^*} |\mathcal{I}^D|(z)}
\]

where \(z^*\) is the point where the image admits its maximal value and \(z \neq z^*\) means that squares of size \(2\lambda \times 2\lambda\) centered at \(z\) and \(z^*\) do not intersect.
Figure: SNR computation versus number of receivers ($B = 0.3 \, \text{Hz}, \, T = 800 \, \text{s}$)
Figure: SNR computation versus time ($J = 21, B = 0.3$ Hz)
Figure: SNR computation versus scaled bandwidth ($J = 61, T = 800$ s)
SNR interpretation

- SNR ratio is linear with respect to number of receivers
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- SNR ratio is linear with respect to number of receivers
- SNR ratio is linear with respect to square root of time

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SNR interpretation

- SNR ratio is linear with respect to number of receivers
- SNR ratio is linear with respect to square root of time
- SNR ratio is linear with respect to square root of bandwidth
- We can summarize these results by

\[ \text{SNR} \sim J \sqrt{BT} \]

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Conclusion and perspectives

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- Exploit higher order cross-correlations to improve SNR?


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- Exploit higher order cross-correlations to improve SNR?
- Perspective: SNR analysis for inhomogeneous media.