Inverse boundary problems for elliptic PDE and best approximation by analytic functions

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Overview

- Boundary value problems  
  Dirichlet, Cauchy

- Normed spaces of generalized analytic functions  
  Hardy

- Application: a physical free boundary problem  
  plasma

- Conclusion
**Conductivity equation**

Let $\Omega \subset \mathbb{R}^2$ with smooth boundary $\Gamma = \partial \Omega$ (Hölder or Dini-smooth)

$\Omega$ simply connected: $\Omega \simeq$ disk $\mathbb{D}$, $\Gamma \simeq$ circle $T$

or annular: $\Omega \simeq A$, $\Gamma \simeq T \cup \varrho T$

(also in multiply connected domains)

Conductivity coefficient $\sigma$ Lipschitz smooth function in $\Omega$ (known)

Consider solutions $u$ to (u):

$$\text{div} \ (\sigma \ \text{grad} \ u) = \text{div} \ (\sigma \ \nabla \ u) = 0 \quad \text{in} \ \Omega$$

(u)

distributional sense $\ 0 < c \leq \sigma \leq C$ \quad second order elliptic equation $\ \Delta u + \nabla (\log \sigma). \nabla u = 0$
Boundary value problems

- Cauchy (inverse) problem: \(|I|, |J| > 0\) partial overdetermined boundary data

Given measures \(u\) and \(\sigma \partial_n u\) on \(I \subset \Gamma\) of a solution \(u\) to \((u)\),

\[
(u): \operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega
\]

recover \(u, \sigma \partial_n u\) on \(J = \Gamma \setminus I\) (or \(\partial_n u\))

\(n\) outer unit normal

\(\sigma\) given

pair of Dirichlet-Neumann data \((\phi_I, \psi_I)\) on \(I\), \(\phi_I \in L^2_{\mathbb{R}}(I), \psi_I \in W^{-1,2}_{\mathbb{R}}(I)\)...

compatibility...

- Dirichlet (direct) problem:

Given measures of \(u\) on \(\Gamma\), recover \(u\) in \(\Omega\) (and \(\sigma \partial_n u\), on \(\Gamma\))

well-posed for Dirichlet data \(\phi \in L^2_{\mathbb{R}}(\Gamma)\)...

(already for smooth data)

\(L^2\) boundary data \(\rightsquigarrow\) smooth conductivity \(\sigma\), tradeoff

practically: pointwise corrupted boundary measurements
\[ \Omega = A \]

\[ l = \mathbb{T}, J = \varrho \mathbb{T} \]

\[ \Omega = \mathbb{D}, I \subset \mathbb{T}, J = \mathbb{T} \setminus I \]
Generalized Cauchy-Riemann equations: for $\Omega = \mathbb{D}$

$$u \text{ solution to } (u): \text{div} \left( \sigma \nabla u \right) = 0 \quad \Rightarrow \exists v \text{ such that in } \Omega:\nabla$$

$$\left\{ \begin{array}{l}
\partial_x v = -\sigma \partial_y u \\
\partial_y v = \sigma \partial_x u
\end{array} \right. \quad \text{whence } \text{div} \left( \frac{1}{\sigma} \nabla v \right) = 0$$

Function $v$: $\sigma$-conjugated to $u$  $\quad v \text{ unique up to additive constant}$

If $u$ solution to (u) and its $\sigma$-conjugated $v$ have $L^2(\Gamma)$ trace, then Cauchy-Riemann equations hold up to boundary $\Gamma$:

$$\partial_\theta v = \sigma \partial_n u \quad \text{ for } \Omega = \mathbb{A}: \exists v \text{ if compatibility boundary condition}$$

$\partial_\theta$ tangential derivative
Generalized analytic functions

In $\Omega \cong \mathbb{D} \subset \mathbb{R}^2 \cong \mathbb{C}$ complex plane

$$X = (x, y) \cong z = x + iy, \quad \partial = \partial_z = \frac{1}{2}(\partial_x - i \partial_y), \quad \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i \partial_y)$$

$u$ solution to $(u)$: $\text{div} (\sigma \nabla u) = 0$

$\nabla \cong \bar{\partial}, \text{div} \cong \text{Re} \partial$

$\Leftrightarrow f = u + i \nu$ satisfies conjugated Beltrami equation

$$\bar{\partial} f = \nu \partial f \quad (f)$$

for $\nu = \frac{1 - \sigma}{1 + \sigma} \in W^{1,\infty}(\Omega), \ |\nu| \leq \kappa < 1$ in $\Omega$

$f$ solution to $(f) \iff u = \text{Re} f$ solution to $(u)$

$(f)$ conformally invariant

$\neq \mathbb{C}$-linear Beltrami equation: $\bar{\partial} g = \nu \partial g$, quasi-conformal map.

$f(z, \bar{z}), u(x, y), \nu(x, y)$ in $\Omega \cong \mathbb{A}$, compatibility condition needed for $\Leftarrow$
Harmonic and analytic functions

Generalization of homogeneous situations $\sigma = \text{cst} \mapsto \sigma = 1$, $\nu = 0$

Holomorphic / complex analytic functions $\bar{\partial}F = 0$ in $\mathbb{D} \subset \mathbb{C}$:

$\Omega = \mathbb{D}$ unit disc or $\Omega \simeq \mathbb{D}$ conformally equivalent

$X = (x, y) \simeq z = x + iy$, $\partial = \partial_z = \frac{1}{2}(\partial_x - i \partial_y)$, $\bar{\partial} = \partial_z = \frac{1}{2}(\partial_x + i \partial_y)$

Laplace operator $\Delta = 4\bar{\partial}\partial = 4\partial\bar{\partial} = \partial_x^2 + \partial_y^2$

$F(z) = \sum_{k \geq 0} \hat{F}_k z^k = \sum_{k \geq 0} \hat{F}_k r^k e^{ik\theta}$, $z = re^{i\theta} \in \mathbb{D}$, $r < 1$

(Fourier series, coefficients $\hat{F}_k$) $\bar{\partial} F = 0$ ($F$ holomorphic) $\iff F = u + iv$

with $\Delta u = 0$ and $\Delta v = 0$: harmonic $u$ and conjugate function $v$ satisfying Cauchy-Riemann equations in $\mathbb{D}$:

$$\begin{cases} 
\partial_x v = -\partial_y u \\
\partial_y v = \partial_x u 
\end{cases}$$
Hardy spaces $H^2$ of analytic functions in $\mathbb{D}$

$H^2(\mathbb{D})$: solutions to $\bar{\partial} F = 0$ in $\mathbb{D}$, $\|F\|_2 < \infty$

$$\|F\|_2^2 = \text{ess sup}_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{k \geq 0} |\hat{F}_k|^2$$

Hilbert space $\subset L^2(\mathbb{D})$

Parseval $p = 2$, also $\Omega = \mathbb{A}$ and Banach $H^p$

$\hookrightarrow L^2$ boundary values on $\mathbb{T}$: $\text{tr} \ H^2(\mathbb{D}) \subset L^2(\mathbb{T})$

$L^2(\mathbb{T}) = \text{tr} \ H^2(\mathbb{D}) \oplus \text{tr} \ H^2,0(\mathbb{C} \setminus \bar{\mathbb{D}})$

$\hookrightarrow$ decomposition, projection $P_+$

$\hookrightarrow$ equivalent boundary $L^2(\mathbb{T})$ norm:

$$\|F\|_2 = \|\text{tr} \ F\|_{L^2(\mathbb{T})}$$

$\hookrightarrow$ Cauchy-Riemann equation in $\bar{\mathbb{D}}$, up to boundary $\mathbb{T}$:

$$F = u + iv, \quad \partial_\theta v = \partial_n u, \quad \partial_n v = -\partial_\theta u \quad \text{tr} \ v = \mathcal{H} \text{tr} \ u$$

also Cauchy integral formula, Poisson kernel, Hilbert-Riesz operator + further properties [Duren, Garnett]

$\hookrightarrow$ results for $\sigma = 1, \nu = 0$, Laplace equations (dimension 2 or 3)
Generalized Hardy space $H^2_{\nu}$

Hilbert space $H^2_{\nu} = H^2_{\nu}(\Omega)$: 

- solutions $f$ to $(f)$ 
- bounded in Hardy norm in $\Omega$

\[ \bar{\partial} f = \nu \bar{\partial} f \text{ in } \Omega \]
\[ \| f \|_2 < \infty \]
\( (\sup \text{ of } L^2 \text{ norms on circles in } \Omega) \)

$H^2_{\nu}$ shares many properties of $H^2 = H^2_0$

Properties of $H^2_\nu$

Generalize those of $H^2$ \( \Omega = \mathbb{D} \) or \( \mathbb{A} \) also $H^p_\nu$ \cite{[BFL,F]}

**Theorem** \cite{BLRR} \( f \in H^2_\nu(\Omega) \)

- $f$ admits a non tangential limit $\text{tr} \ f \in L^2(\Gamma)$ on $\Gamma$
- $\text{tr} \ f = 0$ a.e. on $I \subset \Gamma$, $|I| > 0$ implies that $f \equiv 0$
  if $f \not\equiv 0$, then $\log |\text{tr} \ f| \in L^1(\Gamma)$, and $f$ admits isolated zeroes (+ Blaschke condition)
- $\|\text{tr} \ f\|_{L^2(\Gamma)}$ is equivalent to $\|f\|_2$ on $H^2_\nu(\Omega)$
- Closedness of traces: $\text{tr} \ H^2_\nu(\Omega)$ is closed in $L^2(\Gamma)$
- $\text{Re} \text{tr} \ f = 0$ a.e. on $\Gamma$ implies that $f \equiv 0$ in $\Omega$
  (up to constant)

whenever

$$f \in H^2_{\nu,0}(\Omega) = \{ f \in H^2_\nu(\Omega), \int_\mathbb{T} \text{Im} \text{tr} \ f = 0 \}$$

$\Gamma = \mathbb{T}$ or $\mathbb{T} \cup \partial \mathbb{T}$

+ maximum principle in modulus
Properties of \( \text{tr} \ H^2_\nu(\mathbb{D}) \)

**Corollary [BLRR]**

- \( \forall \phi \in L^2_{\mathbb{R}}(\mathbb{T}), \ \exists! \ f \in H^2_{\nu,0}(\mathbb{D}) \) such that \( \text{Re} \ \text{tr} \ f = \phi \)

  moreover,

  \[
  \| \text{tr} \ f \|_{L^2(\mathbb{T})} \leq c_\nu \| \phi \|_{L^2(\mathbb{T})}
  \]

- conjugation operator \( \mathcal{H}_\nu \) bounded on \( L^2_{\mathbb{R}}(\mathbb{T}) \)

  \[ \text{Re} \ \text{tr} \ f = \phi \quad \xrightarrow{\mathcal{H}_\nu} \quad \text{Im} \ \text{tr} \ f = \mathcal{H}_\nu \phi \]

  \[
  f \in H^2_{\nu,0}(\mathbb{D}) \iff \text{tr} \ f = (I + i\mathcal{H}_\nu)\phi, \ \phi \in L^2_{\mathbb{R}}(\mathbb{T})
  \]

- density:

  let \( I \subset \mathbb{T}, \ J = \mathbb{T} \setminus I \) such that \( |J| > 0 \)

  then, restrictions to \( I \) of functions in \( \text{tr} \ H^2_\nu(\mathbb{D}) \) dense in \( L^2(I) \)

  also in \( H^p_\nu, \ 1 < p < \infty \)
Other situations

- Generalization to $\Omega = A$ annulus

$$A = \mathbb{D} \setminus \varrho \overline{\mathbb{D}}$$

and multiply connected smooth domains [BFL, F]

Dirichlet in $H^2_\nu(A)$ for data in $L^2_{\mathbb{R}}(A) \ominus S$

$$H^2_\nu(A) = \text{solutions to } (f): \bar{\partial} f = \nu \partial f \text{ in } A \text{ with } \|f\|_2 < \infty$$

$$S = \{ \phi \in L^2_{\mathbb{R}}(\partial A) \text{ s.t. } \phi|_T = C, \ \phi|_{\varrho T} = -C, \ C \in \mathbb{R} \}$$

Density of restrictions on $I \subseteq \mathbb{T}$ of $\text{tr } H^2_\nu(A)$ in $L^2(I)$ (or $I \subseteq \varrho \mathbb{T}$)

- Conformal invariance of $(f)$: $\Omega \sim \mathbb{D}$ or $\Omega \sim A$

- For Hölder smooth $\nu \in W^{1,r}(\Omega), \ r > 2$

$$\text{in } H^p_\nu(\Omega) \quad \text{with } \infty > p > r/(r - 1)$$
For related conductivity PDE

\[ u \text{ solution to } (u) \text{ in } \Omega : \]
\[ \text{div} \ (\sigma \nabla u) = 0 \iff u = \text{Re} \, f \text{ with } f \text{ solution to } (f) \text{ in } \Omega \]
\[ \Omega \simeq D \text{ or } A \text{ if } \Omega \simeq D, \iff \]

Dirichlet boundary value problems:
from prescribed boundary data \( \phi \in L^2_{\mathbb{R}}(\Gamma) \)
recover \( u \) in \( \Omega \) solution to \( (u) \) such that \( \text{tr} \ u = \phi \) on \( \Gamma \)

From Dirichlet theorem in \( H^2_{\nu,0}(\Omega) \):
\[ \exists! \ u \text{ in } L^p_{\mathbb{R}}(\Omega) \text{ solution to } (u) \text{ such that } \text{tr} \ u = \phi \]
\[ \text{tr} \ f = \phi + i \int_{\Gamma} \sigma \partial_n u = \phi + i \mathcal{H}_\nu \phi , \quad \| u \|_2 = \| \text{tr} \ u \|_{L^2(\Gamma)} = \| \phi \|_{L^2(\Gamma)} \]

Also, unique continuation properties for \( (u) \)...

bounded conjugation operator \( \rightsquigarrow \) stability properties for \( (u) \)...

Dirichlet-Neumann map: \( \Lambda \phi = \partial_\theta \mathcal{H}_\nu \phi \)
For related conductivity PDE

Cauchy inverse problems, $I \subset \mathbb{T}$

Given $\phi_I$ and $\psi_I$ in $L^2_{\mathbb{R}}(I)$
recover $u$ solution to $(u)$ in $\Omega$ such that $\text{tr } u = \phi_I$, $\sigma \partial_n u = \psi_I$ on $I$

Let

$$\Phi = \phi_I + i \int_I \psi_I \in L^2(I)$$

Density results: $[\text{tr } H^2_{\nu}]|_I$ dense in $L^2(I)$

Runge property (compatible boundary data)

$$\exists f_k \in \text{tr } H^2_{\nu}, \| \Phi - f_k \|_{L^2(I)} \to 0$$

$(k \to \infty)$

either $\Phi \in \text{tr } H^2_{\nu}|_I$ already and $\| \Phi - f_k \|_{L^2(\mathbb{T})} \to 0$

However

or $\Phi \notin \text{tr } H^2_{\nu}|_I$

and $\| f_k \|_{L^2(J)} \to \infty$
For related conductivity PDE

\[ \Phi = \phi_I + i \int_I \psi_I \in L^2(I) \setminus (\text{tr } H^2_{\nu})|_I: \]

\[ \exists u_k = \text{Re } f_k \text{ solution to } (u) \text{ in } \Omega \]
\[ \|\delta_0 H_\nu u_k - \psi_I\|_{L^2(I)} \rightarrow 0 \]
\[ \Rightarrow \text{ Look for } \text{tr } u \simeq \phi_I, \sigma \partial_n u \simeq \psi_I \text{ on } I \text{ with tr } u \text{ bounded on } J... \]
\[ \Rightarrow \text{Bounded extremal problems (BEP) in } \text{tr } H^2_{\nu} \]

best constrained approximation
Best constrained approximation in $H^2_\nu$

Regularization: bounded extremal problems (BEP)

Let $I \subset \Gamma$, $|I|, |J| > 0$, $\varepsilon > 0$

$$\Omega = \mathbb{D}, \Gamma = \mathbb{T}, J = \mathbb{T} \setminus I$$

$$B = \left\{ f \in \text{tr } H^2_\nu, \| \text{Re } f \|_{L^2(J)} \leq \varepsilon \right\} |I| \subset L^2(I).$$

**Theorem** [BFL, FLPS] (BEP) well-posed $\nu = 0$: [BLP]

\forall \text{ function } \Phi \in L^2(I), \exists \text{ unique } f_* \in B \text{ such that}

$$\| \Phi - f_* \|_{L^2(I)} = \min_{f \in B} \| \Phi - f \|_{L^2(I)}$$

Moreover, if $\Phi \notin B$, then $\| \text{Re } f_* \|_{L^2(J)} = \varepsilon$

**Proof:** bounded conjugation, density result

also in $\Omega \simeq \mathbb{A}$, with $I \subset \mathbb{T}$, $J = (\mathbb{T} \setminus I) \cup g\mathbb{T}$

also in $H^p_\nu$, for $L^p(I)$ data, or with other norm constraints
Constructive issues in $H^2_\nu$

Computation algorithm, from $\Phi \in L^2(I)$, $\Omega = \mathbb{D}$, $A \ (I \subseteq \mathbb{T})$ [AP,BFL,FLPS]

$\perp$ projection operator $L^2(\Gamma) \to \text{tr } H^2_\nu;^0$:

$$P_\nu \phi = \frac{1}{2} (\phi + i H_\nu \phi)$$

vanishing mean on $\mathbb{T}$

Solution to (BEP): given $\Phi \in L^2(I)$, $M > 0$ 

Toeplitz-Hankel operators on $H^2_\nu$

$$P_\nu(\chi_I f_*) - \gamma P_\nu(\chi_J f_*) = (I - (\gamma + 1)P_\nu \chi_J) f_* = P_\nu(\Phi \vee 0)$$

for ! Lagrange parameter $\gamma < 0$ s.t. $\|f_*\|_{L^2(J)} = M$

$$\min_{f \in \text{tr } H^2_\nu} \|\Phi - f\|_{L^2(I)} + \gamma \|\text{Re } f\|_{L^2(J)}$$

$\gamma \% M$ smoothly decreasing

Complete families of solutions, for computations in $H^2_\nu(\Omega)$ and $L^2(\Gamma)$

$\leadsto$ Bessel/exponentials, toroidal harmonics (w.r.t. $\sigma$ or $\nu$, and $\Omega$) polynomials?

$\nu = 0$: Fourier basis, polynomials [L.-P.-Pozzi]
Plasma equilibrium model in a tokamak

In 2D poloidal sections, poloidal magnetic flux $u$:

$$\text{div} \left( \frac{1}{\chi} \nabla u \right) = \text{div} (\sigma \nabla u) = 0$$

in the vacuum $\Omega$, conductivity $\sigma = \frac{1}{\chi}$

Maxwell equations, cylindrical coordinates $(x, y) = (R, Z)$, $\phi = $ cte

$$\Omega \simeq A_0 \subset \mathbb{R}^2$$

annular domain between plasma and chamber

$$\Gamma = \Gamma_e \cup \Gamma_p$$

limitor $\Gamma_l \subset \Omega$ inside plasma, Grad-Shafranov equation, control

From pointwise magnetic data on outer boundary $\Gamma_e$ (tg poloidal mag. field)

$$u, \quad B_\rho = -\frac{1}{\chi} \partial_t u, \quad B_t = \frac{1}{\chi} \partial_n u$$

recover plasma boundary $\Gamma_p$

free boundary problem
Plasma in tokamak

\[ \sigma(x, y) = \frac{1}{x} = \frac{2}{z + \bar{z}} \text{ smooth in } \Omega \]

\[ \nu(z, \bar{z}) = \frac{z + \bar{z} - 2}{z + \bar{z} + 2} \]

Complete families of solutions to (u) and (f):
- Bessel-exponentials \( \Omega \sim \mathbb{D}_0 \)
- toroidal harmonics for \( \Omega \sim \mathbb{A}_0 \)

In place of Fourier bases for \( \sigma = 1 \) or \( \nu = 0 \)

More about constructive issues, Toeplitz operators, algorithms \( \sim \text{GT-EP} \)
Toroidal harmonics

Toroidal coordinates
\[ \tau = \log \left( \frac{MA}{MB} \right), \quad \eta = \overline{AMB} \]

Annulus \( \simeq A_0 \) between circles
\[ \tau = \text{cst} \]
Toroidal harmonics

Complete family $\mathcal{T} = (u_j(\tau, \eta))_{j \geq 0}$ in $L^2(\partial A_0)$

$$(\tau = \text{cst})$$ [F]

$$u_j(\tau, \eta) = a \frac{\sinh \tau}{\sqrt{\cosh \tau - \cos \eta}} \left\{ \begin{array}{l}
P_j^{1} \left( \cosh \tau \right) \\
Q_j^{1} \left( \cosh \tau \right)
\end{array} \right\} \left\{ \begin{array}{l}
\cos j \eta \\
\sin j \eta
\end{array} \right\}$$

$$x = \frac{a \sinh \tau}{\cosh \tau - \cos \eta}, \quad y = \frac{a \sin \eta}{\cosh \tau - \cos \eta}$$

solutions to $\text{div} \left( \frac{1}{x} \nabla u_j \right) = 0$

$P_j^{1}, Q_j^{1}$ associated Legendre functions

$\leadsto$ explicit $\sigma$-harmonic conjugate functions $v_j = \mathcal{H}_\nu u_j$

on $P_j^{0}, Q_j^{0}$, $\text{div} (x \nabla v_j) = 0$
Plasma in tokamak

From measurements of \( u, \sigma \partial_n u \) on outer boundary \( \Gamma_e \), find level line \( \Gamma_p \) of associated solution \( u \) to (\( u \)), tangent to limitor \( \Gamma_l \)

Take a first such \( \Gamma_{p,0} \)

Data transmission \( \Gamma_e \rightsquigarrow \Gamma_{p,0} : \)

\( u, \sigma \partial_n u \) on \( I = \Gamma_e \rightsquigarrow u, \sigma \partial_n u \) on \( J_0 = \Gamma_{p,0} \), \( u \) in \( \Omega_0 \)

Cauchy boundary inverse problem in \( \Omega_0 \)

s

solve (BEP)

\( u, B_\rho \rightsquigarrow \phi_I, B_t = \partial_\theta v = \sigma \partial_n u \rightsquigarrow \psi_I \rightsquigarrow \) Cauchy data \( \Phi \) on \( \Gamma_e \)

constraint \( \| \text{Re} f_* - c \|_{L^2(J_0)} \leq M \) small, \( c \) constant

Free boundary problem \( \Gamma_p \):

iterate 1st step \( \rightsquigarrow \Gamma_p \), last closed level line tangent to \( \Gamma_l \)

\( u, \partial_n u \) on \( \Gamma_l \rightsquigarrow \Gamma_{p,1} : \{ u = \max_{\Gamma_l} u \} \) with shape optimization [Fischer-Privat]
Approximation on $\Gamma_e$

Of given smooth data $u$, $\partial_n u$ by toroidal harmonics expansions
Plasma boundary recovery

\[ \Gamma^{(1),10}_p \]
\[ \Gamma^{(1),14}_p \]
\[ \Gamma^{(1),18}_p \]
\[ \Gamma_{EFIT} \]
\[ \Gamma_{LIM} \]
\[ \Gamma_{APOLO} \]
Plasma boundary recovery

Poloidal section of tokamak Tore Supra

Reconstruction of plasma boundary $\Gamma_p$ from measurements $\bigcirc$ of poloidal flux $u$ and $\times$ of magnetic field $\partial_n u$ on $\Gamma_e$ with series of toroidal harmonics (18 terms)
Application to plasma shaping in a tokamak

Tore Supra
(CEA-IRFM Cadarache)

magnetic field $B$, flux $u$
Conclusion

Work in progress:  

- More about generalized Hardy classes $H^p_\nu$  
  
  factorization, operators; $p = 1, \infty$?  
  
  density of traces for $\Omega = \mathbb{A}$;  
  
  reproducing kernel in $H^2_\nu$?  
  
  extremal problems  
  
  solutions $w = e^s F$ to related $\bar{\partial} w = \alpha \bar{w}$  
  
  $\alpha = \bar{\partial} \log \sigma^{1/2}$  

- Other elliptic operators (and relat-ed/-ing PDEs)?  
  
  + time $t$?  
  
  Schrödinger $\Delta w \simeq |\alpha|^2 w + (\partial \alpha) \bar{w}$, 3D Laplace + symmetry properties $\rightarrow$ 2D conductivity ($u$)  

- Stability estimates  
  
  unique continuation for ($u$) and Schrödinger eq.?  

- In higher dimensions?  

- Non smooth conductivity $\sigma$ (or coefficients $\nu, \alpha$)?  
  
  anisotropic (matrix-valued)?  
  
  (up to now, $\mathbb{R}$-valued Hölder smooth $\sigma$, $r > 2$, in $H^p_\nu(\Omega), p > r/(r-1)$)  

  Also, geometrical issues: Bernoulli type (free boundary) problems  

  other tokamaks, ITER: non smooth boundary (X point)
Main references


[BLF] Baratchart, Fischer, Leblond, Dirichlet/Neumann problems and Hardy classes for the planar conductivity equation (subm.)


[FLPS] Fischer, Leblond, Partington, Sincich, BEP in Hardy spaces for the conjugate Beltrami equation in simply conn. dom. (2011)


and Astala, Iwaniec, Martin (2008), Kravchenko (2009), Vekua (1962), ...