Extension of the adjoint method

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Topology optimization formulates a design problem as an optimal material distribution problem.

The search of an optimal domain is equivalent to finding its characteristic function, it is a 0-1 optimization problem.

Different approaches make this problem differentiable:

- Relaxation, homogenization,
- Level set,
- Topological derivatives.
To present the basic idea, let $\Omega$ be a domain of $\mathbb{R}^d$, $d \in \mathbb{N}\setminus\{0\}$ and $j(\Omega) = J(u_\Omega)$ a cost function to be minimized, where $u_\Omega$ is a solution to a given partial differential equation defined in $\Omega$.

Let $x$ be a point in $\Omega$ and $\omega_1$ a smooth open bounded subset in $\mathbb{R}^d$ containing the origin. For a small parameter $\rho > 0$, let $\Omega \setminus \omega_\rho$ be the perturbed domain obtained by making a perforation $\omega_\rho = \rho \omega_1$ around the point $x$.

The **topological asymptotic expansion** of $j(\Omega \setminus \omega_\rho)$ when $\rho$ tends to zero is the following:

$$j(\Omega \setminus \omega_\rho) = j(\Omega) + f(\rho)g(x) + o(f(\rho)).$$

where $f(\rho)$ denotes an explicit positive function going to zero with $\rho$ and $g(x)$ is called the **topological gradient** or **topological derivative**.

It is usually simple to compute and is obtained using the solution of direct and adjoint problems defined on the initial domain.
In topology optimization, there are some **drawbacks of topological derivatives** approaches:

- The asymptotic topological expansion is not easy to obtain for **complex problems**.
- It needs to be **adapted for many particular cases** such as the creation of a hole on the boundary of an existing one or on the original boundary of the domain.
- It is difficult to determine the variation of a cost function when a **hole is to be filled**.
- In real applications of topology optimization, a **finite perturbation** is performed and not an infinitesimal one.
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Consider the following steady state equation

\[ F(c, u) = 0 \text{ in } \Omega, \]

where \( c \) is a distributed parameter in a domain \( \Omega \).

The aim is to minimize a cost function \( j(c) := J(u_c) \) where \( u_c \) is the solution of the direct equation for a given \( c \).

Let us suppose that every term is differentiable. We are considering a perturbation \( \delta c \) of the parameter \( c \).

The direct equation can be seen as a constraint, and as a consequence, the Lagrangian is considered:

\[ \mathcal{L}(c, u, p) = J(u) + (F(c, u), p), \]

where \( p \) is a Lagrange multiplier and \((\cdot, \cdot)\) denotes the scalar product in a well-chosen Hilbert space.
To compute the derivative of $j$, one can remark that $j(c) = \mathcal{L}(c, u_c, p)$ for all $c$, if $u_c$ is the solution of the direct equation. The derivative of $j$ is then equal to the derivative of $\mathcal{L}$ with respect to $c$:

$$d_c j(c) \delta c = \partial_c \mathcal{L}(c, u_c, p) \delta c + \partial_u \mathcal{L}(c, u_c, p) \partial_c u \delta c.$$ 

All these terms can be calculated easily, except $\partial_c u \delta c$, the solution of the linearized problem:

$$\partial_u F(c, u_c)(\partial_c u \delta c) = -\partial_c F(c, u_c) \delta c.$$ 

To avoid the resolution of this equation for each $\delta c$, the term $\partial_u \mathcal{L}(c, u_c, p)$ is cancelled by solving the following adjoint equation. Let $p_c$ be the solution of the adjoint equation:

$$\partial_u F(c, u_c)^T p_c = -\partial_u J^T.$$ 

So the derivative of $j$ is explicitly given by

$$d_c j(c) \delta c = \partial_c \mathcal{L}(c, u_c, p_c) \delta c.$$
Note that if the Lagrangians $\mathcal{L}(c + \delta c, \ldots, \ldots)$ and $\mathcal{L}(c, \ldots, \ldots)$ are defined on the same space, we have

$$j(c + \delta c) - j(c) = \mathcal{L}(c + \delta c, u_{c+\delta c}, p_c) - \mathcal{L}(c, u_c, p_c) = (\mathcal{L}(c + \delta c, u_{c+\delta c}, p_c) - \mathcal{L}(c + \delta c, u_c, p_c)) + (\mathcal{L}(c + \delta c, u_c, p_c) - \mathcal{L}(c, u_c, p_c)).$$

In the case of a regular perturbation $\delta c$, the second term gives the main variation and the first term is of higher order.

In the case of a singular perturbation, the first term is of the same order as the second one and cannot be ignored. Then the variation of $u_c$ has to be estimated.

The basic idea of the numerical vault is to update the solution $u_c$ by solving a local problem defined in a small domain around $x_0$. 


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In the linear case is studied, consider the variational problem depending on a parameter $\varepsilon$:

$$a^\varepsilon(u^\varepsilon, v) = \ell^\varepsilon(v) \quad \forall v \in \mathcal{V}^\varepsilon,$$

where $\mathcal{V}^\varepsilon$ is a Hilbert space, $a^\varepsilon$ is a bilinear, continuous and coercive form and $\ell^\varepsilon$ is a linear and continuous form.

Typically, $\mathcal{V}^\varepsilon$ est tel que $H_0^1 \subset \mathcal{V}^\varepsilon \subset H^1$.

The aim is to minimize a cost function which depends of $\varepsilon$.

$$j(\varepsilon) := J^\varepsilon(u^\varepsilon).$$

The cost function $J^\varepsilon$ is of class $C^1$, the adjoint problem associated to the problem is

$$a^\varepsilon(w, p^\varepsilon) = -\partial_u J^\varepsilon(u^\varepsilon)w \quad \forall w \in \mathcal{V}^\varepsilon,$$

where $p^\varepsilon$ is the solution of this problem.
Suppose that $a^\varepsilon$, $\ell^\varepsilon$ and $J^\varepsilon$ are integrals over a domain $\Omega$.

The domain $\Omega$ is split into two parts, a part $D$ containing the perturbation, and its complementary $\Omega_0 = \Omega \setminus D$.

The forms $a^\varepsilon$, $\ell^\varepsilon$, and the cost function $J^\varepsilon$ are decomposed in the following way:

- $a^\varepsilon = a_{\Omega_0} + a_D^\varepsilon$,
- $\ell^\varepsilon = \ell_{\Omega_0} + \ell_D^\varepsilon$,
- $J^\varepsilon = J_{\Omega_0} + J_D^\varepsilon$,

where $a_{\Omega_0}$, $l_{\Omega_0}$ et $J_{\Omega_0}$ are independants of $\varepsilon$.

- $\mathcal{V}_{\Omega_0}$, the space consisting of functions of $\mathcal{V}^\varepsilon$ and $\mathcal{V}^0$ restricted to $\Omega_0$,
- $\mathcal{V}_D^\varepsilon$, the space consisting of functions of $\mathcal{V}^\varepsilon$ restricted to $D$,
- $\mathcal{V}_D^0$, the space consisting of functions of $\mathcal{V}^0$ restricted to $D$, \[ 
We assume that $\mathcal{V}^0 \subset \mathcal{V}^\varepsilon$.

Let us consider $u_0^\varepsilon$, the local update of $u^0$:

$$
\begin{cases}
\text{Find } u_D^\varepsilon \in \mathcal{V}_D^\varepsilon \\ a_D^\varepsilon(u_D^\varepsilon, v) = \ell_D^\varepsilon(v), & \forall v \in \mathcal{V}^\varepsilon_D, 0, \\
u_D^\varepsilon = u^0 \text{ on } \partial D.
\end{cases}
$$

The update of $u^0$, named $\tilde{u}^\varepsilon$, is given by:

$$
\begin{cases}
\tilde{u}^\varepsilon = \begin{cases}
u_D^\varepsilon & \text{in } D, \\
u^0 & \text{in } \Omega_0.
\end{cases}
\end{cases}
$$
Hypotheses:
There exist three positive constants $\eta$, $C$ and $C_u$ independent of $\varepsilon$ and a positive real valued function $f$ defined on $\mathbb{R}_+$ such that

$$\lim_{\varepsilon \to 0} f(\varepsilon) = 0,$$

$$\|J^\varepsilon(v) - J^\varepsilon(u) - \partial_uJ^\varepsilon(u)(v - u)\|_{\mathcal{V}_\varepsilon} \leq C\|v - u\|_{\mathcal{V}_\varepsilon}^2, \quad \forall v, u \in B(u^0, \eta),$$

$$\|u^\varepsilon - u^0\|_{\mathcal{V}_{\Omega_0}} \leq C_{u}f(\varepsilon),$$

$$\lim_{\varepsilon \to 0} \|p^\varepsilon - p^0\|_{\mathcal{V}_\varepsilon} = 0.$$

**Proposition**

*Under these hypotheses, we have $\|u^\varepsilon - \tilde{u}^\varepsilon\|_{\mathcal{V}_\varepsilon} = \mathcal{O}(f(\varepsilon))$.***

**Theorem (Update of the direct solution)**

*Under these hypotheses, we have

$$j(\varepsilon) - j(0) = \mathcal{L}^\varepsilon(\tilde{u}^\varepsilon, p^0) - \mathcal{L}^0(u^0, p^0) + o(f(\varepsilon)).$$*
\( \mathcal{V}^0 \) is not necessary a sub-space of \( \mathcal{V}^\varepsilon \).

The definition of \( \tilde{u}^\varepsilon \) stays the same.

Let us consider \( p_D^\varepsilon \), the local update of \( p^0 \)

\[
\begin{align*}
\left\{ \begin{array}{l}
\text{Find } p_D^\varepsilon \in \mathcal{V}_D^\varepsilon \text{ solution of} \\
a_D^\varepsilon (w, p_D^\varepsilon) = -\partial_u J_D^\varepsilon (u_D^\varepsilon) w, \\
p_D^\varepsilon = p^0 \quad \forall w \in \mathcal{V}_{D,0}^\varepsilon, \\
\end{array} \right. \\
\text{on } \partial D.
\end{align*}
\]

The update of \( p^0 \), named \( \tilde{p}^\varepsilon \), is given by:

\[
\tilde{p}^\varepsilon = \left\{ \begin{array}{ll}
p_D^\varepsilon & \text{in } D, \\
p^0 & \text{in } \Omega_0.
\end{array} \right.
\]
Hypotheses:
There exist four positive constants $\eta$, $C$, $C_u$ and $C_p$ independent of $\varepsilon$ and a positive real valued function $f$ defined on $\mathbb{R}_+$ such that

$$\lim_{\varepsilon \to 0} f(\varepsilon) = 0,$$

$$\|J^\varepsilon(v) - J^\varepsilon(u) - \partial_u J^\varepsilon(u)(v - u)\|_{\mathcal{V}^\varepsilon} \leq C\|v - u\|_{\mathcal{V}^\varepsilon}^2, \forall v, u \in B(u^0, \eta),$$

$$\|u^\varepsilon - u^0\|_{\mathcal{V}_{\Omega_0}} \leq C_u f(\varepsilon),$$

$$\|p^\varepsilon - p^0\|_{\mathcal{V}_{\Omega_0}} \leq C_p f(\varepsilon).$$

**Proposition**

*Under these hypotheses, we have $\|p^\varepsilon - \tilde{p}^\varepsilon\|_{\mathcal{V}^\varepsilon} = O(f(\varepsilon)).$*

**Theorem (Update of the direct and adjoint solutions)**

*Under these hypotheses, we have*

$$j(\varepsilon) - j(0) = \mathcal{J}^\varepsilon(\bar{u}^\varepsilon, \bar{p}^\varepsilon) - \mathcal{J}^0(u^0, p^0) + O(f(\varepsilon)^2).$$
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Let \( \Omega \) be a rectangular bounded domain of \( \mathbb{R}^2 \) and \( \Gamma \) be its boundary, composed of two parts \( \Gamma_1 \) and \( \Gamma_2 \). The points of the rectangle are submitted to a vertical displacement \( u \) solution of the following equation:

\[
\begin{cases}
- \nabla . \sigma(u) = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma_1, \\
\sigma(u)n = \mu & \text{on } \Gamma_2,
\end{cases}
\]

where \( \phi(u) = \frac{1}{2}(Du + Du^T) \), \( \sigma(u) = hH_0\phi(u) \), \( \sigma(u) \) is the stress distribution, \( H_0 \) is the Hooke tensor and \( h(x) \) represents the material stiffness.

The optimization problem is to minimize the following cost function:

\[
J(h) = \int_{\Gamma_2} g.u \, dx,
\]
From up to down, the perturbations and the associated curves
- blue-circle : exact variation,
- red-asterisk : estimation obtained by the numerical vault,
- green-square : estimation obtained by the adjoint method.
The abscissa is the inhomogeneity stiffness.
The stiffness varies from 0 to 2.5 with the material stiffness equal to 1.
The ordinate is the variation of the cost function.
From up to down, the perturbations and the associated curves
- blue-circle : exact variation,
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The abscissa is the inhomogeneity stiffness.
The stiffness varies from 0 to 2.5 with the material stiffness equal to 1.
The ordinate is the variation of the cost function.
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From up to down, the perturbations and the associated curves
- **blue-circle**: exact variation,
- **red-asterisk**: estimation obtained by the numerical vault,
- **green-square**: estimation obtained by the adjoint method.

The abscissa is the inhomogeneity stiffness.
The stiffness varies from 0 to 1 with the material stiffness equal to 1.
The ordinate is the variation of the cost function.
Let $\Omega$ be a rectangular bounded domain of $\mathbb{R}^2$ and $\Gamma$ be its boundary, composed of two parts $\Gamma_1$ and $\Gamma_2$. The points of the rectangle are submitted to a horizontal displacement $u$ solution of the following equation:

$$
\begin{align*}
- \nabla \cdot \sigma(u) &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma_1, \\
\sigma(u)n &= \mu \quad \text{on } \Gamma_2,
\end{align*}
$$

where $\phi(u) = \frac{1}{2} (Du + Du^T)$, $\sigma(u) = hH_0\phi(u)$, $\sigma(u)$ is a stress distribution, $H_0$ is the Hooke tensor and $h(x) = 1$ or $\rho$ represents the material stiffness. The optimization problem is to minimize the following cost function:

$$
J(h) = \int_{\Omega} |u_h - u_{obs}|^2 \, dx.
$$
From left to right, in the first row, the horizontal displacements for the uniform material and the perturbed material. In the second row, the inclusion and the detection obtained with the numerical vault.
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Creation of a new method to solve some difficulties of topological derivatives.

Major advantages of the method are that the numerical vault is non invasive and can be used in different kinds of problems with a parallel computing implementation. Our method could be a tool in theoretical investigations.


**Perspectives**
- coupling with multiscale algorithms,
- semilinear equations.
Thank you for your attention